Description Operators in Partial Propositional Type Theory

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A Type Theory containing $\{T, F\}$ as only basic type is Leśniewski's *Protothetics* [2]. Henkin [3] arrived at this formal system independently of Leśniewski, presenting a (complete) calculus for Propositional Types using only λ and identity as primitive symbols. The completeness proof lies in the fact that it is possible to have a *name* in the object language for every element of the hierarchy of types (this hierarchy is obtained by passing from any two sets $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ to a set $\mathcal{D}_{\alpha\beta}$). Since the elements of the domains above $\{T, F\}$ —above $\mathcal{D}_t$— are functions, the addition of an *undefined value* to $\mathcal{D}_t$ naturally produces functions which are undefined when applied to certain arguments (i.e., functions for which we can compute that there is no result). Therefore, a 3-valued Propositional Type Theory seems to be a suitable vehicle for dealing with *partial functions*. A semantics for such a logic was given by Lepage [5] and Lapierre [4].

However, not every function from $\{T, F, \circledast\}$ to $\{T, F, \circledast\}$ should be considered as a partial function. The behavior of the undefined truth-value has to be compatible with any increase in information, that is, it must be *regular*. Lepage [5], who found inspiration in Kleene's undefined value, restricted the set of partial functions in $\mathcal{D}^{\circledast}_{tt}$ to those that are monotonic with respect to the following partial order (on the new basic type): $\circledast \leq \circledast$, $\circledast \leq T$, $\circledast \leq F$, $T \leq T$ and $F \leq F$. Monotonicity is imposed to any domain $\mathcal{D}_{\alpha\beta}$.

It is worth recalling at this point that the set of monotonic functions from $\mathcal{D}^{\circledast}_t$ to $\mathcal{D}^{\circledast}_{tt}$ has many interesting functions: in particular, *all* Kleene's strong connectives. In fact, the name for conjunction in Henkin's hierarchy is now the name for Kleene's strong conjunction (see [6, p. 33]), provided that identity is interpreted in a different manner (see [4, p. 533]). The names for the remaining Kleene's connectives involve conjunction and negation. Lepage's 3-valued logic also distinguishes between total and non-total objects, which are defined inductively (see [4, p. 527]), and includes a function symbol $\mathfrak{J}(A_\alpha)$ saying that "the interpretation of A_α is a total object".

Lepage [6] offers a sound axiomatic system for this 3-valued Propositional Type Theory, but its completeness is posed as an open question. The lack of proof is due to the "impossibility of having a canonical name in the object language for every partial function without modifying the theoretical framework in an essential way" (Lepage [6, p. 37]).

The aim of this paper is to define description operators for Lepage's system (the first step towards a Nameability Theorem) by means of suitable modifications and exploiting the semantics of Kleene's strong connectives. Firstly, we will explain why Henkin's description operator for type $\langle tt\rangle$ cannot be simply transferred to the domain $\mathcal{D}^{\circledast}_{tt}$ of the 3-valued hierarchy. Then, we will define an election function à la Henkin (see [3, p. 328]) of type $\langle \alpha t\rangle\alpha$, proving that such a function *is* in the 3-valued hierarchy, i.e., its monotonicity. We will also show that the denotation of the expression

$$\iota_{\langle tt\rangle t} := (\lambda f_{tt}(\mathfrak{J}(f_{tt}) \rightarrow f_{tt} \equiv (\lambda x_t x_t)))$$

is the election function for $\mathcal{D}^{\divideontimes}_{tt}$ (where $\rightarrow$ is Kleene's strong conditional). Finally, we will outline how to prove that, for any α, there is a closed expression $\iota_{\langle \alpha t \rangle \alpha}$ such that $(\iota_{\langle \alpha t \rangle \alpha})^d = \mathbf{t}^{(\alpha)}$ where $\mathbf{t}^{(\alpha)}$ is the election function for the arbitrary type α. To state this result, we have to (1) add a proper symbol U_α (of type α) to the set of primitive symbols of Propositional Type Theory and (2) build a "lambda hierarchy of undefinedness" whose interpretation is, for each level of the hierarchy, (3) the *least defined element* of the corresponding domain. The following definitions are introduced:

Definition 1 (Lambda hierarchy of undefinedness). We inductively define U_σ for any type σ:

1. For $\sigma = t$, $U_\sigma \coloneqq U_t$.
2. For $\sigma = \alpha\beta$, $U_\sigma \coloneqq \lambda x_\alpha U_\beta$.

Definition 2 (Least defined element). We inductively define $\mathbf{u}_\sigma$ for any type σ:

1. For $\sigma = t$, $\mathbf{u}_\sigma \coloneqq \divideontimes$.
2. For $\sigma = \alpha\beta$, $\mathbf{u}_\sigma$ is the function of $\mathcal{D}_{\alpha\beta}$ such that, for any $\theta \in \mathcal{D}_\alpha$, $\mathbf{u}_{\alpha\beta}(\theta) = \mathbf{u}_\beta$.

We stipulate that the interpretation of any element U_σ of the lambda hierarchy (which found inspiration in [1]) is $\mathbf{u}_\sigma$. Hence, the undefined value, and any function of type $\alpha\beta$ sending every element of $\mathcal{D}^{\divideontimes}_\alpha$ to $\mathbf{u}_\beta$, will finally have a name in the object language. From a purely philosophical point of view, the fact that we can designate functions that are completely undefined may be problematic. What kind of *object* is a function that is undefined for every element of its domain? Should we even call it "partial function"? We think of these functions as *extreme cases* that serve us as denotation for those expressions pointing at non-existent functions of the 3-valued hierarchy. Let us conclude by explaining this perspective in some detail.

Lepage [6, p. 37] explicitly rejected the possibility of including a name for the third truth-value in the object language, for the reason that "one immediate consequence would be that, even for partial total valuations, some expressions would remain undefined". However, we claim that, once $\{U_\alpha\}_{\alpha \in \mathsf{PT}}$ (where PT is the set of type symbols) has been added to the set of primitive symbols, and having defined the lambda hierarchy of undefinedness, it holds for any meaningful expression A_α that its interpretation is in $\mathcal{D}_\alpha$.

To sum up, the talk will present several syntactic and semantic modifications of the Partial Propositional Type Theory developed in [5], [4] and [6], which is nothing but a Kleene's logic. Although the most recent research on Henkin's axiomatic system does not focus on making it many-valued (see, among others, [7] and [8]), our goal is to show that these modifications allow us to define description operators even with a third truth-value in play.

References

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