The 4th Homotopy Group of the 3-Sphere in Cubical Agda

Axel Ljungström and Anders Mörtberg

Stockholm University, Stockholm, Sweden
{axel.ljungstrom, anders.mortberg}@math.su.se

One of the most extensive works on synthetic homotopy theory in HoTT/UF so far can be found in the PhD thesis of Brunerie [1]. The main result of the thesis is that $\pi_4(\mathbb{S}^3)$, the fourth homotopy group of the 3-sphere, is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. For many years, this result has remained unformalised. The two main problems seem to have been:

1. Some theorems/constructions have simply been very hard to formalise, despite having relatively detailed proofs in Brunerie's thesis.
2. Many results in the latter half of the thesis concerning cohomology rely on theorems concerning the so called *smash product (of two types)*. In particular, they rely on the associativity of the smash product – a result whose proof is mostly omitted in Brunerie's thesis.

In this talk, we will present a full formalisation[1] of Brunerie's theorem carried out in Cubical Agda [5]. Our solutions to the aforementioned problems can be summarised as follows.

1. Streamline some of the proofs – in particular one, concerning an application of the so called James construction.
2. Use the cohomology theory presented in [2] in order to escape the smash product induced coherence hell.

We will present our formalisation with an emphasis on the mathematical changes to Brunerie's proof. To this end, we will give a brief overview of the main ideas of Brunerie's original proof and pay special attention to those details which have been simplified in our work. The proof is divided into two parts.

Part 1: Constructing the Brunerie number. In the first part, Brunerie constructs what is now known as the *Brunerie number*, an integer $\beta : \mathbb{Z}$ such that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$. To this end, he defines two maps

$$W : \mathbb{S}^3 \to \mathbb{S}^2 \vee \mathbb{S}^2 \qquad\qquad \nabla : \mathbb{S}^2 \vee \mathbb{S}^2 \to \mathbb{S}^2$$

where $\vee$ denotes the wedge sum [4, Section 6.8]. The first half of Brunerie's thesis is concerned with constructing a chain of isomorphisms $\pi_4(\mathbb{S}^3) \cong \pi_3(J_2(\mathbb{S}^2)) \cong \pi_3(\mathsf{cofib}(\nabla \circ W))$, where $J_2(\mathbb{S}^2)$ and $\mathsf{cofib}(\nabla \circ W)$ respectively are the pushouts of the spans $\mathbb{S}^2 \times \mathbb{S}^2 \hookleftarrow \mathbb{S}^2 \vee \mathbb{S}^2 \to \mathbb{S}^2$ and $1 \longleftarrow \mathbb{S}^3 \xrightarrow{\nabla \circ W} \mathbb{S}^2$. Brunerie then applies the *Blakers-Massey theorem* [3] and the long exact sequence of homotopy groups [4, Theorem 8.4.6] to obtain an exact sequence

$$\underbrace{\pi_3(\mathbb{S}^3)}_{\cong \langle e \rangle} \xrightarrow{\pi_3(\nabla \circ W)} \underbrace{\pi_3(\mathbb{S}^2)}_{\cong \langle h \rangle} \longrightarrow \pi_3(\mathsf{cofib}(\nabla \circ W)) \longrightarrow 1$$

where $e : \pi_3(\mathbb{S}^3)$ and $h : \pi_3(\mathbb{S}^2)$ are generators of the homotopy groups. Using the above sequence, the Brunerie number may be defined as follows.

[1] See https://github.com/agda/cubical/blob/master/Cubical/Homotopy/Group/Pi4S3/Summary.agda

Definition/Theorem 1. *Define* $\eta := \pi_3(\nabla \circ W)(e) : \pi_3(\mathbb{S}^2)$*. The Brunerie number* β *is the unique integer satisfying* $\eta = \beta \cdot h$*. In particular, it satisfies* $\pi_3(\mathsf{cofib}(\nabla \circ W)) \cong \pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$

We have formalised the above result in Cubical Agda. In the formalisation, we deviate in one major way. The crucial step in the above proof is the isomorphism $\pi_4(\mathbb{S}^3) \cong \pi_3(J_2(\mathbb{S}^2))$. In Brunerie's thesis, this result uses the *James construction* [1, Chapter 3]. In our formalisation, however, we have been able to avoid the general James construction altogether. Instead, we construct the isomorphism directly, via the following lemma.

Lemma 2. *There is a (non-trivial) family of equivalences* $F : \mathbb{S}^2 \to \left\|J_2(\mathbb{S}^2)\right\|_3 \simeq \left\|J_2(\mathbb{S}^2)\right\|_3$ *such that* $F(*_{\mathbb{S}^2})$ *is the identity equivalence.*

Theorem 3. $\Omega \left\|\mathbb{S}^3\right\|_4 \simeq \left\|J_2(\mathbb{S}^2)\right\|_3$*. Hence, we also have* $\pi_4(\mathbb{S}^3) \cong \pi_3(J_2(\mathbb{S}^2))$*.*

Proof. Via Lemma 2 and the *encode-decode* method [4, Section 8.9]. □

The above theorem allows us to skip most of pages 67-81 in Brunerie's thesis and were relatively straightforward to formalise in Cubical Agda. The price we pay is, of course, that the theory we develop is less general.

Part 2: Proving that $|\beta| = 2$**.** For this step, we need to show for an isomorphism of our choice $\psi : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ that $|\psi(\eta)| = 2$. There is a canonical one, coming from the long exact sequence of homotopy groups, but its action of η is rather unclear. Instead, Brunerie uses the so called *Hopf Invariant*, $\mathsf{HI} : \pi_3(\mathbb{S}^2) \to \mathbb{Z}$. He then proves the following three facts.

Theorem 4. *The Hopf invariant is a group homomorphism.*

Theorem 5. *For the generator* $h : \pi_3(\mathbb{S}^2)$*, we have* $|\mathsf{HI}(h)| = 1$*. Hence, the Hopf invariant is an isomorphism.*

Theorem 6. *We have* $\mathsf{HI}(\eta) = \pm 2$*, and hence* $|\beta| = 2$*.*

Theorem 4–6 all rely on various results concerning the cup product $\smile : H^n(A) \to H^m(A) \to H^{n+m}(A)$, where $H^n(A)$ is the *nth integral cohomology group of a type A* (see e.g. [2, Chapter 3]). In particular, they rely on the following result.

Theorem 7. *The cup product is associative, distributive and graded-commutative.*

In Brunerie's thesis, this theorem is stated, but its proof relies partly on an omitted proof of the associativity of the smash product. Theorem 7 *was*, however, proved and formalised by the authors and Brunerie in [2, Section 4.2], using an alternative definition of the cup product. This has allowed us to continue and complete the formalisation of Brunerie's original proof. This includes, apart from Theorems 4–6, constructions like the (iterated) Hopf construction and the Gysin sequence [1, Chapter 6]. On the way, we found some new "baby Brunerie numbers" which, like the Brunerie number, are integers definable in Cubical Agda, but which we are not able to normalise.

As is often the case, the formalisation caught some minor typos is Brunerie's thesis (e.g. the definition of $W_{A,B}$ on page 82 is ill-typed in the push case). However, on the whole, we found the proofs to be correct and "formalisation ready".

References

[1] Guillaume Brunerie. *On the homotopy groups of spheres in homotopy type theory.* PhD thesis, Université Nice Sophia Antipolis, 2016.

[2] Guillaume Brunerie, Axel Ljungström, and Anders Mörtberg. Synthetic integral cohomology in cubical agda. In Florin Manea and Alex Simpson, editors, *30th EACSL Annual Conference on Computer Science Logic, CSL 2022, February 14-19, 2022, Göttingen, Germany (Virtual Conference)*, volume 216 of *LIPIcs*, pages 11:1–11:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

[3] Kuen-Bang Hou (Favonia), Eric Finster, Daniel R. Licata, and Peter LeFanu Lumsdaine. A Mechanization of the Blakers-Massey Connectivity Theorem in Homotopy Type Theory. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '16, pages 565–574, New York, NY, USA, 2016. ACM.

[4] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Self-published, Institute for Advanced Study, 2013.

[5] Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types. *Proceedings of the ACM on Programming Languages*, 3(ICFP):87:1–87:29, August 2019.