Synthetic Tait Computability for Simplicial Type Theory

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1 Introduction

Riehl and Shulman [13] introduced a simplicial extension of (homotopy) type theory to reason synthetically about $(\infty,1)$-categories. Indeed, the semantics of this theory matches up with established results from homotopy theory, *i.e.*, the synthetic $(\infty,1)$-categories in the theory correspond externally to internal $(\infty,1)$-categories in an arbitrary given ∞-topos (implemented as complete Segal objects), cf. [10, 15, 23]. However, certain meta-theoretic properties of this simplicial type theory (STT), such as canonicity and normalization, have not been investigated yet.

In this work, we adapt the framework of *synthetic Tait computability (STC)* due to Sterling and Harper [18] to simplicial type theory, following the original work on cubical type theory (CTT) from Sterling–Angiuli [17] and Sterling [16]. The framework has previously been used to give syntax-invariant proofs of canonicity (cf. [14]) and normalization for cubical type theory, generalizing and internalizing previous accounts of normalization by evaluation (NBE) using the internal language of a sufficiently structured topos.

Contribution We explain how to define analogously a simplicial version of STC giving rise to a notion of computability topos for simplicial type theory. We'll also report on our progress in establishing a normalization proof à la [17, 16] for STT in this setting.

2 Simplicial Type Theory

Simplicial type theory (STT) due to [13] augments traditional Martin–Löf type theory by two features: (i) two additional pre-type layers (for directed *cubes* and (sub-)*shapes*), and (ii) *extension types*. The latter had been previously devised by Lumsdaine–Shulman in unpublished work. They can be understood as Π-*types with strict side conditions*: Given a shape inclusion $\Phi \hookrightarrow \Psi$, a type family $A : \Psi \to \mathcal{U}$ and a partial section $\sigma : \prod_{t:\Phi} A(t)$, the *extension type* $\left\langle \prod_\Psi A\big|^{\Phi}_{\sigma} \right\rangle$ can be understood as the type of all *totalizations of* σ up to judgmental equality, *i.e.*, all sections $\sigma' : \prod_{t:\Psi} A(t)$ such that $\sigma(t) \equiv \sigma'(t)$ for $t : \Phi$. This is also familiar from cubical type theory where path types are defined in a similar way. The rules of extension types are analogous to the ones for Π-types, but containing additional definitional equalities, cf. [13].

With this at hand, in STT one can then define a notion of weak composition of morphisms for a type A by requiring that the map $A^{\Delta^2} \to A^{\Lambda^2_1}$ induced by the inclusion of the shape $(\bullet \to \bullet \to \bullet)$ into the filled 2-simplex be a weak equivalence (*Segal condition*, cf. [11, 7]). This is a crucial ingredient for defining a synthetic notion of $(\infty,1)$-category.

Building on fundamental parts of synthetic higher category theory developed in [13] there has been work on synthetic $(\infty, 1)$-categories in the simplicial setting [12, 4, 23, 9] and in the bicubical setting [22]. Kudasov is working on a prototype proof assistant supporting STT [8].

3 Simplicial Synthetic Tait Computability

Presentation as a fibered signature In his recent PhD thesis [16], Sterling develops a logical framework to define a variety of type theories. The idea is to present a type theory by a *signature*, which specifies abstractly the potential judgments to be formed. The actual admissible contexts of the type theory are organized into a *category of atomic contexts* which arises as a submodel of the syntactic model.

An abstract *signature* $\mathbb{S}$ is given by its collection of concrete *implementations* $U : \mathbb{S}$. The collection of signatures forms a category **SIG** whose morphisms are "functions" $U : \mathbb{S} \to \mathbb{T}$, $x \mapsto U(x)$. In fact, the signatures give rise to a type theory with dependent sums, products, identity types, and a terminal type $\mathbb{1}$. This implies that the category **SIG** is finitely complete.

Because of its multi-layered structure, STT is presented as a chain of projection maps from appropriate signatures:

$$\mathbb{STT} \xrightarrow{\pi_1} \mathbb{T}\mathrm{ope} \xrightarrow{\pi_0} \mathbb{C}\mathrm{ube}$$

Simplicial STC and computability topos We adapt the axiomatization from [17] to the setting of simplicial rather than cubical type theory. One notable difference is that the simplicial interval $\mathbb{2}$ is *not* tiny, *i.e.*, exponentiation $(-)^{\mathbb{2}} : \mathcal{E} \to \mathcal{E}$ does not have a right adjoint.

Namely, we devise an axiomatization of an ambient category $\mathcal{E}$ whose internal language supports an appropriate notion of computability structure. This can be instantiated by a topos pushout (actually a gluing presheaf topos) of an open (syntactic) with a closed (semantic) subtopos, using an appropriate simplicial *figure shape*, as in [17, 16].

Towards normalization of STT Along the lines of [17, 16] we can then carry out a version of normalization by evaluation (NBE), based on an analogous notion of *stabilized neutrals* to capture the (more general) case of extension types.

In the classical picture of NBE, after Tait [21], one constructs for a type A a chain of maps (*reflection* and *reification*)

$$\mathrm{ne}(A) \to [\![A]\!] \to \mathrm{nf}(A)$$

from the *neutral terms* of type A to the (computational) semantics of terms of type A, and from there to the *normal forms* of type A (cf. also [5, 2, 20, 6, 1, 3, 19]).

For CTT, it was the insight of Sterling–Angiuli that the conditions for a term to be neutral could not be formulated reasonably. However, they were able to instead capture the conditions for a term to *cease* to be neutral. Informally, if p denotes a path term and i an interval variable, the neutral path application term $p(i)$ would cease to be neutral in case $i \equiv 0$ or $i \equiv 1$ (since this would force another computation). This gives rise to a modified version of Tait's method, called *stabilized* Tait yoga.

We are re-using this idea for the case of STT where this *frontier of instability* for application $\tau(t)$ of a section $\tau : \left\langle \prod_{\Psi} A \big|^{\Phi}_{\sigma} \right\rangle$ to a tope variable $t : \Psi$ depends on t satisfing the condition $\varphi(t)$ defining the distinguished subshape Φ.

With those modifications, our current progress indicates that the methods by Sterling–Angiuli and Sterling to prove normalization carry over well to the simplicial setting.[1] We will report on our progress in this program.

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[1] Note that, in contrast to [13], we do not assume any axioms in the present version of simplicial type theory such as function extensionality (neither for ordinary function nor extension types), let alone univalence, as these would obstruct canonicity.

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