# Semantics for two-dimensional type theory

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## 1 Introduction

In recent years, efforts have been made to develop *directed* type theory. Roughly, directed type theory should correspond to Martin-Löf type theory (MLTT) as $\infty$-categories correspond to $\infty$-groupoids. Besides theoretical interest in directed type theory, it is hoped that such a type theory can serve as a framework for synthetic directed homotopy theory and synthetic $\infty$-category theory. Applications of those, in turn, include reasoning about concurrent processes [2].

Several proposals for *syntax* for directed type theory have been given (reviewed in Section 2), but are ad-hoc and are not always semantically justified. The *semantic* aspects of directed type theory are particularly underdeveloped; a general notion of model of a directed type theory is still lacking.

In this work, we approach the development of directed type theory from the semantic side. We introduce *comprehension bicategories* as a suitable mathematical structure for higher-dimensional (directed) type theory. Comprehension bicategories capture several different specific mathematical structures that have previously been used to interpret higher-dimensional or directed type theory.

From comprehension bicategories, we extract the core syntax—judgment forms and structural inference rules—of a two-dimensional dependent type theory that can accommodate directed type theory. We also give a soundness proof of our structural rules. In future work, we will equip our syntax and semantics with a system of variances and type and term formers for directed type theory.

A preprint describing this work in more detail is available on the arXiv [1].

## 2 Related Work

We review only some related work here; an in-depth review of related work is given in [1].

In [7], Licata and Harper developed a two-dimensional dependent type theory with a judgment for *equivalences* $\Gamma \vdash \alpha : M \simeq_A N$ between terms $M, N : A$. These equivalences are postulated to have (strict) inverses. The authors give an interpretation of types as groupoids.

Licata and Harper [6] (see also [5, Chapter 7]) also designed a *directed* two-dimensional type theory and gave an interpretation for it in the strict 2-category of categories. Their syntax has a judgment for *substitutions* between contexts, written $\Gamma \vdash \theta : \Delta$, and *transformations* between parallel substitutions. An important aspect of their work is *variance* of contexts/types, built into the judgments.

Nuyts [9] attempts to generalize the treatment of variance by Licata and Harper, and designs a directed type theory with additional variances, in particular, isovariance and invariance. Nuyts does not provide any interpretation of their syntax, and thus no proof of (relative) consistency.

North [8] develops a type former for *directed* types of morphisms, resulting in a typal higher-dimensional directed type theory based on the judgments of MLTT. The model given by North is in the 2-category of categories, similar to Licata and Harper's [6].

Shulman, in unfinished work [10], aims to develop 2-categorical logic, including a two-dimensional notion of topos and a suitable internal language for such toposes. Our work is similar to Shulman's in the sense that both start from a (bi)categorical notion and extract a language from it, with the goal of developing a precise correspondence between extensions of the syntax and additional structure on the semantics.

Garner [3] studies a typal two-dimensional type theory in the style of Martin-Löf. They add rules that turn any identity type into a discrete type, effectively "truncating" intensional Martin-Löf type theory at 1-types. Garner defines a notion of two-dimensional model based on (strict) *comprehension 2-categories*. Exploiting the restriction to 1-truncated types, they then give a sound and complete interpretation of their two-dimensional type theory in any model.

# 3 Details

We introduce a notion of "model" of two-dimensional type theory that is a quite straightforward generalization of the 1-categorical comprehension categories introduced by Jacobs [4].

**Definition 1.** A **comprehension bicategory** is given by a *strictly* commuting diagram of pseudofunctors

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{\chi} & \mathsf{B}^{\to} \\ & {\scriptstyle \mathsf{p}}\searrow \quad \swarrow{\scriptstyle \mathsf{cod}} & \\ & \mathsf{B} & \end{array}$$

where $\mathsf{p}$ is equipped with a global cleaving and a local opcleaving (modelling substitution in a suitable sense), opcartesian 2-cells of $\mathsf{p}$ are preserved under left and right whiskering, and $\chi$ preserves cartesian 1-cells and opcartesian 2-cells.

Examples of comprehension bicategories are plentiful; in particular, taking $\mathsf{B} \equiv \mathsf{Cat}$ and $\mathsf{E} \equiv \mathsf{OpFib}$ yields a structure that is similar to, but crucially not the same as, the structure in which Licata and Harper's interpretation [6] takes place. (A comparison is given in [1, §7.6].)

From the definition of comprehension bicategories we extract a core type theory called BTT. We then prove

**Theorem 1** (Soundness)**.** *We can interpret* BTT *in any comprehension bicategory.*

Our type theory can be simplified in lockstep with the categorical structure in which it is interpreted; we give various pairs of simplifications in [1, §7.5].

# 4 Conclusion

Our work is very general; it allows for the modelling of the structural rules of previous suggestions for directed type theory. Furthermore, it can be used as a framework for defining and studying more specialized syntax and semantics, in lockstep.

In separate work we are going to extend our structural rules with variances and a suitable hom-type former à la North [8].

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