Pre-bilattices in Univalent Foundations

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Abstract

Bilattices are algebraic structures used in logic and artificial intelligence, comprising two lattice orders (usually the one modelling amount of truth and the other modelling amount of information) as well as some property and/or operator that links the two orders. This paper sums up ongoing work on how bilattices can be defined in Univalent Foundations, in particular in the UniMath Coq library.

1 Introduction

1.1 Bilattices

Pre-bilattices are sets equipped with two lattice orderings, usually aimed to model simultaneously the validity of, and degree of knowledge about, sentences from a logical language. They have found applications in diverse fields, among which in multi-valued logics [3, 13], paraconsistent reasoning [2, 3], and logic programming [7, 1], and they have also been studied from an algebraic perspective [5, 10, 6, 9].

In the literature, there are two main ways that the lattices of a pre-bilattice are connected:

- Via a property that includes operations from both orders; the two properties that are usually studied are interlacing and distributivity, which will be described below, but also modularity has been considered [11].
- Via an extra operator, usually negation (a negation on the truth order that is monotonic on the knowledge order) or conflation (a negation on the knowledge order that is monotonic on the truth order). Pre-bilattices with a negation and/or a conflation operator are called bilattices. Also more generic negation-like operators [14] and residuated bilattices [8] have been studied.

Here, we will consider interlacing and distributivity, leaving negation operators (i.e. bilattices) for future work. This is mainly because the most mathematically interesting basic results, the main one being the representation theorem which will be discussed below, are meaningful even at the level of pre-bilattices.

1.2 Univalent Foundations and UniMath

UniMath [16] is a library of formalised mathematics built on the Coq theorem prover [15], using Homotopy Type Theory [12] as its foundation. As such, it's a constructive structural framework in which it is possible to formalise mathematical entities and proofs. Some algebraic structures have already been formalised in UniMath, among them the notion of lattices; in fact, most properties of lattices that will be needed for pre-bilattices are already proven in UniMath.

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2 Pre-bilattices in UniMath

The type of lattices in UniMath is a dependent type $\texttt{Lattice} : \mathsf{Set} \to \mathcal{U}$, specifying that a lattice has two associative and commutative binary operators, min ($\sqcap$) and max ($\sqcup$), such that $x \sqcap (x \sqcup y) = x$ and $x \sqcup (x \sqcap y) = x$. Moreover, UniMath defines a partial order ($\leq$) for each lattice, by $x \leq y :\equiv (x \sqcap y = x)$ (a mere proposition, since lattices are defined over sets).

For each $X : \mathsf{Set}$, it is hence possible to define the type of *pre-bilattices*,

$$\texttt{PreBilattice}(\texttt{X}) :\equiv \texttt{Lattice}(\texttt{X}) \times \texttt{Lattice}(\texttt{X}),$$

where the first lattice will be considered the *truth* lattice (with min $\wedge_b$, max $\vee_b$, and ordering $\leq_{\mathrm{t}b}$) and the second lattice will be considered the *knowledge* lattice (with min $\otimes_b$, max $\oplus_b$, and ordering $\leq_{\mathrm{k}b}$).

Notice that there are quite a few dualities in place when studying pre-bilattices: the opposite order of a lattice is again a lattice and, moreover, one can swap the truth and knowledge lattices of a pre-bilattice to obtain another pre-bilattice. As a consequence, there emerges a “prove one, get many” situation, which can be utilised in UniMath to facilitate proving properties of pre-bilattices.

The type of *interlaced* pre-bilattices over a set X is

$$\texttt{InterlacedPreBilattice}(\texttt{X}) :\equiv \sum_{b:\texttt{PreBilattice}(\texttt{X})} \texttt{IsInterlaced}(\texttt{b}),$$

where $\texttt{IsInterlaced}(\texttt{b})$ describes that $\wedge_b$ and $\vee_b$ are monotonic (for their first argument) with respect to $\leq_{\mathrm{k}b}$ and that $\otimes_b$ and $\oplus_b$ are monotonic (for their first argument) with respect to $\leq_{\mathrm{t}b}$.

Similarly, one can define the type of *distributive* pre-bilattices, which are those pre-bilattices such that all pairs of $\wedge_b$, $\vee_b$, $\otimes_b$, and $\oplus_b$ are distributive, i.e. $x \star (y \cdot z) = (x \star y) \cdot (x \star z)$, and prove that distributive pre-bilattices are interlaced.

It is also possible, for each pair $l_1 : \texttt{Lattice}(\texttt{X}_1)$, $l_2 : \texttt{Lattice}(\texttt{X}_2)$ of lattices, to define the product pre-bilattice (for its definition, see for example [4, Definition 3.1]) and prove that it is unique up to equivalence (again, due to the fact that lattices are defined over sets).

The representation theorem. A well-known and important result in the theory of bilattices is the representation theorem, stating that product pre-bilattices are interlaced pre-bilattices and vice-versa. The left-to-right part of the proof follows easily from basic properties. The converse can also be proved in UniMath, formalising the proof in [4, Section 3.1]: it involves defining two equivalence relations on the interlaced pre-bilattice, whose equivalence classes will be the lattices forming the product pre-bilattice. To the author's knowledge, some of the mechanisms in UniMath that are necessary for reasoning with equivalence classes (`setquotunivprop` and the lemmas that depend on it) result in non-computable opaque terms, which implies that the representation theorem will be usable for proofs but not for computations; however, the importance of the theorem lies exactly on the fact that it facilitates proving properties of interlaced pre-bilattices by reducing them to product pre-bilattices.

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