The Patch Frame of a Spectral Locale in Univalent Type Theory

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Summary. Stone locales together with continuous maps form a coreflective subcategory of spectral locales and perfect maps. A proof in the internal language of an elementary topos is given in [1, 2]. This proof can be easily translated to univalent type theory using *resizing axioms*. In this work, we show how to achieve such a translation *without* resizing axioms, by working with large, locally small frames with small bases. This turns out to be nontrivial and involves predicative reformulations of several fundamental concepts of locale theory.

Foundations and notation. We work in the context of intensional Martin-Löf type theory without full univalence but with propositional and function extensionality and propositional truncation. We assume the existence of $\sum$ and $\prod$ types as well as the inductive types of $\mathbf{0}$, $\mathbf{1}$, the natural numbers, and lists. We define the type of families on a given type A as $\mathsf{Fam}_{\mathcal{W}}(A) :\equiv \sum_{I:\mathcal{W}} I \to A$. Given a family $(I, \alpha) : \mathsf{Fam}_{\mathcal{W}}(A)$, we often use the abbrevation $\{\alpha(i)\}_{i:I}$ instead of the tuple notation. Given a family $\mathcal{J} :\equiv (J, \beta)$ on the index type I, we write $\{\alpha(j) \mid j \in \mathcal{J}\}$ to denote the composite $(J, \alpha \circ \beta)$ i.e. the subfamily of $\{\alpha(i)\}_{i:I}$ given by (J, β).

Spectral and Stone locales. A *spectral locale* (also called *coherent* [4, pg. 63]) is a locale in which the compact opens form a basis closed under finite meets. A continuous map of spectral locales is perfect iff its defining frame homomorphism preserves compact opens. A *Stone locale* is one that is compact and zero-dimensional (i.e. whose clopens form a basis). Every Stone locale is spectral since the clopens coincide with the compact opens in Stone locales. We denote by **Stone** the category of Stone locales with continuous maps, and by **Spec** the category of Spectral locales with perfect maps. The right adjoint to the inclusion **Stone** $\hookrightarrow$ **Spec** is denoted by **Patch**.

Patch as the frame of Scott-continuous nuclei. A nucleus on a frame is a finite-meet-preserving closure operator. The patch frame of a spectral frame can be formulated as the frame of Scott-continuous nuclei on the frame [1]. A consequence of this description is that the patch frame freely adds Boolean complements to the given frame.

Locales with small bases. A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame is a type $A : \mathcal{U}$ equipped with (1) a partial order $- \leq - : A \to A \to \Omega_{\mathcal{V}}$, a top element $\mathbf{1} : A$, (2) a binary meet operation $- \wedge - : A \to A \to A$, and (3) a join operation $\bigvee - : \mathsf{Fam}_{\mathcal{W}}(A) \to A$ such that binary meets distribute over arbitrary joins:

$$\prod_{x:A} \prod_{(I,\alpha):\mathsf{Fam}_{\mathcal{W}}(A)} x \wedge \bigvee(I, \alpha) = \bigvee(I, \lambda i.\ x \wedge \alpha(i)).$$

We follow the standard convention of talking about locales, for which we use the variables $X, Y, Z, \ldots$, and referring to their frame of opens as $\mathcal{O}(X)$. A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locale X is said to have a *small basis* iff there exists a $\mathcal{W}$-family $\{B_i\}_{i:I}$ on $\mathcal{O}(X)$ that satisfies:

$$\mathsf{isBasisFor}\,(\mathcal{B}, X) \quad :\equiv \quad \prod_{U:\mathcal{O}(X)} \sum_{J:\mathsf{Fam}_{\mathcal{W}}(I)} U \text{ is the least upper bound of } \{B_j \mid j \in J\}.$$

Spectrality, regularity, and zero-dimensionality. The notions of spectral, regular, and zero-dimensional locales are defined, in the impredicative setting of set theory, as locales in which certain kinds of opens form bases. A spectral locale, for example, is one in which the set of compact opens forms a basis closed under finite meets. Such definitions are problematic in a predicative context as it is not always the case that such sets of opens are small. We therefore restrict attention to locales with small bases and express these notions by imposing conditions on the bases in consideration. The notion of a spectral $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locale, for example, is defined as a locale with a small basis $\{B_i\}_{i:I}$ together with the requirements:

$$\prod_{i:I} B_i \text{ compact} \quad \text{and} \quad \prod_{i,j:I} \left\| \sum_{k:I} B_i \wedge B_j = B_k \right\|.$$

Same idea is employed in the definitions of regular and zero-dimensional locales. Another contribution concerns the predicative reformulations of locale-theoretic results about these notions, suitable for formalisation in type theory.

Construction of the patch frame. The strategy of [1] is to start from the known fact that the set of nuclei on a frame themselves form a frame, and then conclude that the Scott-continuous nuclei form a subframe. However, the first step does not seem to be available in our setting, and we need a different method of proof to show that the Scott-continuous nuclei form a frame. Similarly, other constructions in [1, 2] need to be completely rethought.

Open nuclei and AFT. In the impredicative setting, to any open U of a locale X, there is an associated open nucleus $\neg`U' :\equiv V \mapsto U \Rightarrow V$, where $- \Rightarrow -$ denotes Heyting implication, which is of fundamental importance in the construction of the patch. In the absence of resizing axioms, however, it does not seem to be possible in univalent type theory to construct Heyting implication for an arbitrary frame. Nevertheless, this is possible for locally small frames with small bases. More generally, we prove the Adjoint Functor Theorem for such frames in type theory: given a $(\mathcal{U}, \mathcal{V}, \mathcal{V})$-locale X with a small basis and a $(\mathcal{U}', \mathcal{V}, \mathcal{V})$-locale (not necessarily with a small basis), a monotone map $\mathcal{O}(Y) \to \mathcal{O}(X)$ has a right adjoint iff it preserves all joins of $\mathcal{O}(Y)$.

Formalisation. Most of our results have been formalised using the Agda proof assistant as part of the first author's `formal-topology-in-UF` library [5]. A more up-to-date formalisation is being developed as part of the second author's `TypeTopology` [3] library. Crucial components have already been formalised in modules `Frame`, `CompactRegular`, `GaloisConnection`, `AdjointFunctorTheoremForFrames`, and `HeytingImplication` of `TypeTopology`[1].

References

[1] Martín H. Escardó. "On the Compact-regular Coreflection of a Stably Compact Locale". In: *Electronic Notes in Theoretical Computer Science* 20 (1999), pp. 213–228. ISSN: 15710661. DOI: `10.1016/S1571-0661(04)80076-8`.

[2] Martín H. Escardó. "The Regular Locally Compact Coreflection of a Stably Locally Compact Locale". In: *Journal of Pure and Applied Algebra* 157.1 (Mar. 8, 2001), pp. 41–55. ISSN: 0022-4049. DOI: `10.1016/S0022-4049(99)00172-3`.

[1]Can be viewed on the `master` branch of `TypeTopology` at `https://github.com/martinescardo/TypeTopology`.

[3] Martín H. Escardó and contributors. *TypeTopology*. Agda library. url: https://github.com/martinescardo/TypeTopology.

[4] Peter T. Johnstone. *Stone Spaces*. Cambridge: Cambridge Univ. Press, 2002. isbn: 978-0-521-33779-3.

[5] Ayberk Tosun. *formal-topology-in-UF*. Agda library. Dec. 24, 2021. url: https://github.com/ayberkt/formal-topology-in-UF.