BioTT: a Family of Brouwerian Intuitionistic Theories Open to Classical Reasoning

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1 Introduction

One difference between Brouwerian intuitionistic logic and classical logic is their treatment of time. In classical logic truth is atemporal, whereas in intuitionistic logic it is time-relative. An example of a time-relative notion is that of Brouwer's choice sequences, which are finite sequences of entities (e.g., natural numbers) that are never complete, and can always be further extended with new choices [16; 5; 25; 26; 19; 27; 22]. This manifestation of the evolving concept of time in intuitionistic logic entails a notion of computability that goes far beyond that of Church-Turing [11, Sec.5]. Brouwer used this concept among other things to define the continuum [4, Ch.3], and it has further been used in so-called intuitionistic (weak) counterexamples to derive the negation of classical axioms such as the Law of Excluded Middle (LEM) [15; 9; 20; 17].

We have built an intuitionistic extensional type theory called BITT [7] (see `https://github.com/vrahli/NuprlInCoq/tree/beth` for its Coq formalization), which features choice sequences and is given meaning through a Beth model [28; 6; 14, Sec.145; 12, Sec.5.4; 11]. We have showed that this theory is anti-classical following the counterexamples mentioned above. This is not only due to the presence of choice sequences, but also to its Beth model, which provides an anti-classical notion of time, which forces some properties on choice sequences to be undecidable. We subsequently developed another intuitionistic extensional type theory called OpenTT [8] (see `https://github.com/vrahli/NuprlInCoq/tree/ls3/` for its Coq formalization), which is given meaning through a "relaxed" notion of a Beth model, called the *open bar* model, which sufficiently weakens the "undecided" nature of choice sequences to enable validating classical axioms, such as LEM. OpenTT also features choice sequences, and includes standard choice sequence axioms, namely: the Axiom of Open Data, a density axiom, and a discreteness axiom [23; 24; 18; 11].

These two theories and models share a common core, which forms the basis for a family of extensional type theories with choice sequences, that can either be made anti-classical or classical-compatible depending on the chosen model. This family provides a computational setting for exploring the implications of time-relative constructs such as choice sequences, For example, it can enable the development of constructive Brouwerian real number theories. It also provides a mean to capture a more relaxed notion of time, providing a basis for more classically-inclined Brouwerian intuitionistic theories. Let us now describe this family of theories at a high level, which we call BioTT here for *Brouwerian Intuitionistic Open Type Theories*.

2 World-Based Calculus

BioTT relies on a untyped call-by-name λ-calculus, whose core syntax includes:

$$v \in \texttt{Value} ::= vt \mid \star \mid \underline{n} \mid \upsilon \mid \lambda x.t \mid \texttt{inl}(t) \mid \texttt{inr}(t) \mid \langle t_1, t_2 \rangle$$
$$vt \in \texttt{Type} ::= \mathbb{N} \mid t_1 < t_2 \mid \mathbb{U}_i \mid \mathbf{\Pi} x{:}t_1.t_2 \mid \mathbf{\Sigma} x{:}t_1.t_2 \mid \{x : t_1 \mid t_2\} \mid t_1{+}t_2 \mid t_1 = t_2 \in t \mid \texttt{Free}$$
$$t \in \texttt{Term} ::= x \mid v \mid t_1\ t_2 \mid \texttt{let}\ x, y = t_1\ \texttt{in}\ t_2 \mid \texttt{fix}(t) \mid \texttt{case}\ t\ \texttt{of}\ \texttt{inl}(x) \Rightarrow t_1\ \texttt{|}\ \texttt{inr}(y) \Rightarrow t_2$$

with numbers $\underline{n}$ as primitives, injections, pairs, where x is a variable, and where υ is a choice sequence name, which inhabit the type `Free` of *free* choice sequences (see [7; 8] for further details). In BioTT, a choice sequence is implemented as a *name*, which allows referring to the sequence in computations (see below), along with its current state, which is a list of choices, e.g., a list of numbers for a choice sequence of numbers.

BioTT's core small-step call-by-name operational semantics allows in particular (1) β-reducing applications of λ-abstractions; (2) unrolling fixpoints, (3) examining choice sequences; (4) destructing pairs; and (5-6) destructing injections:

(1) $(\lambda x.t)\ u \mapsto_w t[x\backslash u]$
(2) $\texttt{fix}(v) \mapsto_w v\ \texttt{fix}(v)$
(3) $\upsilon(\underline{n}) \mapsto_w w[\upsilon][\underline{n}]$
(4) $\texttt{let}\ x, y = \langle t_1, t_2\rangle\ \texttt{in}\ t \mapsto_w t[x\backslash t_1; y\backslash t_2]$
(5) $\texttt{case inl}(t)\ \texttt{of inl}(x) \Rightarrow t_1 \mid \texttt{inr}(y) \Rightarrow t_2 \mapsto_w t_1[x\backslash t]$
(6) $\texttt{case inr}(t)\ \texttt{of inl}(x) \Rightarrow t_1 \mid \texttt{inr}(y) \Rightarrow t_2 \mapsto_w t_2[y\backslash t]$

Note that this semantics is parameterized by a *world* w. BioTT is parameterized by a Kripke frame [21; 20] consisting of a set of *worlds* $\mathcal{W}$ equipped with a reflexive and transitive binary relation $\sqsubseteq$. To support computing with choice sequences, worlds allow storing choice sequences, and the above semantics allows applying the name of a choice sequence υ to a number $\underline{n}$ in order to access the $\underline{n}^{th}$ choice made for υ in the current world w, written $w[\upsilon][\underline{n}]$.

BioTT's core inference rules include standard sequent calculus rules such as the following $\mathbf{\Pi}$-introduction rule, where H is a list of hypotheses (see [7; 8] for details): if $H, z : A \vdash b : B[x\backslash z]$ and $H \vdash A \in \mathbb{U}_i$ then $H \vdash \lambda z.b : \mathbf{\Pi} x{:}A.B$.

3 Bar-Based Forcing Semantics

BioTT is given meaning through a family of forcing interpretations, where types are interpreted as Partial Equivalence Relations [1; 2; 10; 3], which are parameterized by a bar notion B. A bar b is a set of world extending (w.r.t. $\sqsubseteq$) a given world w (we write b_w to indicate the world w that b *bars*), and a bar "notion" is then a predicate B on bars. This interpretation is inductively-recursively [13] defined as (1) an inductive relation $w \vDash T_1 \equiv T_2$ that expresses type equality; and (2) a recursive function $w \vDash t_1 \equiv t_2 \in T$ that expresses equality in a type. In particular, this interpretation is closed under bars as follows:

$$w \vDash T_1 \equiv T_2 \iff \exists b_w. \forall w' \in b_w. \exists T_1', T_2'.(T_1 \Downarrow_{w'} T_1' \wedge T_2 \Downarrow_{w'} T_2' \wedge w' \vDash T_1' \equiv T_2')$$
$$w \vDash t_1 \equiv t_2 \in T \iff \exists b_w. \forall w' \in b_w. \exists T'.(T \Downarrow_{w'} T' \wedge w' \vDash t_1 \equiv t_2 \in T')$$

where $T \Downarrow_w T'$ states that T computes to T' in all extensions (w.r.t. $\sqsubseteq$) of w.

We can then show that according to this interpretation, BioTT forms a type system, in the sense that $w \vDash T_1 \equiv T_2$ and $w \vDash t_1 \equiv t_2 \in T$ are symmetric, transitive, and respect computation, and are also monotonic and local as expected for such possible-world semantics [28; 14; 12, Sec.5.4]. In particular, this is true when the predicate B specifies a topological space of bars.

Beth Bars. B can be instantiated so as to capture Beth bars as follows. A bar b of a world w is a Beth bar iff for all infinite chains of extensions $w \sqsubseteq w_1 \sqsubseteq w_2 \sqsubseteq \dots$, there exists an $i \in \mathbb{N}$ such that $w_i \in b$. The resulting model allows validating the following axioms [7]:

- density: $\mathbf{\Pi} n{:}\mathbb{N}.\mathbf{\Pi} f{:}\mathcal{B}_n.\downarrow\mathbf{\Sigma} a{:}\texttt{Free}.f = a \in \mathcal{B}_n$
- discreteness: $\mathbf{\Pi} a, b{:}\texttt{Free}.(a = b \in \mathcal{B}) + \neg(a = b \in \mathcal{B})$
- ¬LEM: $\neg\mathbf{\Pi} P{:}\mathbb{U}_i.\downarrow(P + \neg P)$

where $\mathbb{N}_n := \{k : \mathbb{N} \mid k < n\}$; $\mathcal{B} := \mathbb{N} \to \mathbb{N}$; $\mathcal{B}_n := \mathbb{N}_n \to \mathbb{N}$; $\texttt{True} := \underline{0} = \underline{0} \in \mathbb{N}$; $\texttt{False} := \underline{0} = \underline{1} \in \mathbb{N}$; $\neg T := T \to \texttt{False}$; and $\downarrow T := \{x : \texttt{True} \mid T\}$.

Open Bars. B can be instantiated so as to capture open bars as follows. A bar b of a world w is an open bar iff for all $w_1 \sqsupseteq w$, there exists a $w_2 \sqsupseteq w_1$ such that for all $w_3 \sqsupseteq w_2$, $w_3 \in b$. The resulting model allows validating the following axioms [8]:

- open data: $\mathbf{\Pi}\alpha{:}\texttt{Free}.P(\alpha) \to \downarrow\mathbf{\Sigma} n{:}\mathbb{N}.\mathbf{\Pi}\beta{:}\texttt{Free}.(\alpha = \beta \in \mathcal{B}_n \to \downarrow P(\beta))$
- density: $\mathbf{\Pi} n{:}\mathbb{N}.\mathbf{\Pi} f{:}\mathcal{B}_n.\downarrow\mathbf{\Sigma} a{:}\texttt{Free}.f = a \in \mathcal{B}_n$
- discreteness: $\mathbf{\Pi} a, b{:}\texttt{Free}.(a = b \in \mathcal{B}) + \neg(a = b \in \mathcal{B})$
- LEM: $\mathbf{\Pi} P{:}\mathbb{U}_i.\downarrow(P + \neg P)$

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