Proof-relevant normalization for intersection types with profunctors

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Abstract

We present a bicategorical orthogonality construction connecting the idempotent and non-idempotent intersection typing systems for profunctors introduced by Olimpieri. It allows us to translate normalization properties between them by providing explicit reduction 2-cells between the interpretation of terms.

In recent years, there was a growing interest in the categorification of models of computation by replacing semantics where types are sets or preorders with richer categorical structures providing finer mathematical invariants. When using bidimensional categorical structures to model computations, rewriting steps between terms become 2-cells carrying information on reductions between programs. We are interested in recent development on intersection typing systems in this direction: Olimpieri considered the bicategory of profunctors equipped with appropriate monads encoding categorifications of non-idempotent and idempotent intersection typing systems [6, 7]. The bidimensional setting offers a proof-relevant framework as the interpretation of a term contains the set of all its derivations.

We use an orthogonality construction between the two models to reduce normalization for the idempotent case to the non-idempotent one by connecting the interpretation of terms in the two settings with explicit reduction 2-cells. We start by recalling the 1-categorical construction by Ehrhard which we generalize to the bicategorical setting. In the 1-categorical case, there is a combinatorial proof for head normalization by extracting an upper bound on the number of head reductions to its normal form [1]. This construction can be extended to the bicategorical setting where Olimpieri further exhibits a reduction isomorphism between the interpretation of a term and its head normal form for profunctors with the non-idempotent monad [6, 7]. On the other hand, the proof uses Tait's reducibility argument for the idempotent case and we lose the reduction witness. One of the applications of our construction is that we now obtain a witness 2-cell for the idempotent setting as well whereas in the 1-dimensional setting, the orthogonality constuction gives no information on the reduction beyond existence.

In order to reduce the proof of normalization for the idempotent case to the combinatorial one for the 1-categorical case, Ehrhard uses an orthogonality connecting the two models of linear logic **Rel** (the category of sets and relations) and **ScottL** (the category of preorders and ideal relations) [3, 2]. For a preorder $P = (|P|, \leq_P)$ and two subsets $x, x' \subseteq |P|$, the orthogonality is defined as follows

$$x \perp_P x' \quad :\Leftrightarrow \quad (x \cap x' \neq \emptyset \Leftrightarrow \downarrow(x) \cap \uparrow(x') \neq \emptyset).$$

It induces a Galois connection: for a subset $X \subseteq \mathcal{P}(|P|)$, its orthogonal $X^\perp$ is given by the set $\{x' \subseteq |P| \mid \forall x \in X, x \perp_P x'\}$. Ehrhard defines a new category **Pop** (preorders with projections) whose objects are pairs (P, X) of a preorder P and a subset $X \subseteq \mathcal{P}(|P|)$ such that $X = X^{\perp\perp}$. Intuitively, X contains all the denotations x of terms in **Rel** that can be mapped to **ScottL** by taking their downclosure $\downarrow(x)$. This new category allows to prove that the interpretation of a term is not empty in **Rel** if and only if it is not empty in **ScottL** and this equivalence is crucially used to translate the combinatorial normalization theorem from the non-idempotent intersection typing system to the idempotent one [2].

In the categorified setting, we replace sets by groupoids and preorders by categories so that the operation mapping a preorder to its underlying set $P \mapsto |P|$ now corresponds to taking the *core* of category (for a small category $\mathbb{A}$, its *core*, denoted by $\mathbf{c}\,\mathbb{A}$, is the maximal subgroupoid of $\mathbb{A}$). Relations $R \subseteq A \times B$ (or equivalently functions $A \times B \to \{0, 1\}$) become profunctors $P : \mathbb{A} \nrightarrow \mathbb{B}$ (or equivalently functors $\mathbb{A} \times \mathbb{B}^{\mathrm{op}} \to \mathbf{Set}$). To represent intersection types, we use finite sequences instead of finite multisets with the free symmetric strict monoidal completion $\mathcal{S}\mathbb{A}$ and the free coproduct completion $\mathcal{C}\mathbb{A}$ for a category $\mathbb{A}$.

Definition 0.1. For a small category $\mathbb{A}$, define $\mathcal{C}\mathbb{A}$ to be the category whose objects are finite sequences $\langle a_1, \dots, a_n\rangle$ of objects of $\mathbb{A}$ and a morphism between two sequences $\langle a_1, \dots, a_n\rangle$ and $\langle b_1, \dots, b_m\rangle$ consists of a pair $(\sigma, (f_i)_{i\in\underline{n}})$ of a function $\sigma : \underline{n} \to \underline{m}$ and a family of morphisms $f_i : a_i \to b_{\sigma(i)}$ in $\mathbb{A}$ for $i \in \underline{n}$. The category $\mathcal{S}\mathbb{A}$ has the same objects but morphisms $(\sigma, (f_i)_{i\in\underline{n}})$ are restricted to the ones where σ is a bijection.

In $\mathcal{S}\mathbb{A}$ there are no morphisms between sequences of different lengths which encodes non-idempotency whereas $\mathcal{C}\mathbb{A}$ allows for duplication and erasure and is used for the idempotent case. In our setting, we consider two bicategories $\mathbf{ProfG}_{\mathcal{S}}$ and $\mathbf{Prof}_{\mathcal{C}}$ representing the categorifications of **Rel** and **ScottL** respectively [4, 5]. The objects of $\mathbf{ProfG}_{\mathcal{S}}$ are small groupoids, the morphisms are profunctors of the form $\mathcal{S}\mathbb{A} \nrightarrow \mathbb{B}$ and the 2-cells are natural transformations between them whereas the objects of $\mathbf{Prof}_{\mathcal{C}}$ are small categories, the morphisms are profunctors of the form $\mathcal{C}\mathbb{A} \nrightarrow \mathbb{B}$ and the 2-cells are natural transformations. Olimpieri studied the idempotent and non-idempotent intersection typing systems associated to the bicategories $\mathbf{Prof}_{\mathcal{C}}$ and $\mathbf{ProfG}_{\mathcal{S}}$ respectively [6, 7].

Similarly to the 1-categorical case, we connect the interpretation of terms in these two bicategories using an orthogonality construction. For a category $\mathbb{A}$ and profunctors $X : \mathbf{1} \nrightarrow \mathbf{c}\,\mathbb{A}$, $X' : \mathbf{c}\,\mathbb{A} \nrightarrow \mathbf{1}$, the orthogonality carries a 2-cell retraction $r : X' \downarrow_{\mathbb{A}} X \Rightarrow X'X$ which intuitively provides a witness connecting the idempotent setting and the non-idempotent setting. The Galois connection in the 1-categorical case now becomes a contravariant adjunction between slice categories and the objects of the new bicategory that we consider are the fixpoints of this adjunction (corresponding to closure under double-orthogonality). We show that we obtain a cartesian closed bicategory and solve fixpoint equations which allows us to obtain explicit retraction 2-cells connecting the interpretations in the idempotent and non-idempotent cases as desired. Since our construction provides us with explicit reduction 2-cells for the idempotent setting, we aim to study execution time by translating existing results in the non-idempotent case [1].

Another advantage of this construction already highlighted by Ehrhard in the 1-categorical setting is its modularity. We can encode various calculi (PCF, standard λ-calculus, bang calculus, call-by-value λ-calculus, etc.) by considering solutions of different fixed point equations. For example, Ehrhard considers a call-by-value calculus by solving the equation $D = !D \multimap !D$ whereas Olimpieri considers retractions $D \lhd !D \multimap D$. If we want to prove normalization for the induced idempotent intersection typing systems, we need to use reducibility candidates for each of the fixpoint equations we consider. The orthogonality construction provides a more modular framework as we know that the result holds for any fixpoint equation obtained using linear logic connectives. In the proof-relevant bicategorical setting, we further automatically have the reduction 2-cell connecting the idempotent and non-idempotent interpretations for all fixpoint equations. Our end goal is also to develop a theory of orthogonality for bidimensional structures enabling us to control interactions between terms and environments and also to restrict the allowed reductions between terms.

References

[1] Daniel de Carvalho. *Sémantiques de la logique linéaire et temps de calcul*. PhD dissertation, Université Aix-Marseille 2, 2008.

[2] Thomas Ehrhard. Collapsing non-idempotent intersection types. *Leibniz International Proceedings in Informatics, LIPIcs*, 16, 09 2012.

[3] Thomas Ehrhard. The Scott model of Linear Logic is the extensional collapse of its relational model. *Theoretical Computer Science*, 424:20–45, 2012. 26 pages.

[4] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. The cartesian closed bicategory of generalised species of structures. *J. Lond. Math. Soc. (2)*, 77(1):203–220, 2008.

[5] Zeinab Galal. A profunctorial Scott semantics. In *5th International Conference on Formal Structures for Computation and Deduction (FSCD 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[6] Federico Olimpieri. *Intersection types and resource calculi in the denotational semantics of lambda-calculus*. PhD thesis, Aix-Marseille, 2020.

[7] Federico Olimpieri. Intersection type distributors. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–15. IEEE, 2021.