Transpension: The Right Adjoint to the Pi-Type

Andreas Nuyts[1] and Dominique Devriese[1]

imec-DistriNet, KU Leuven, Belgium

Presheaf models of dependent type theory [Hof97, HS97] have been successfully applied to model HoTT [BCH14, CMS20, CCHM17, Hub16, KLV12, Ort18, OP18], parametricity [AGJ14, BCM15, ND18a, NVD17], and directed, guarded [BM20] and nominal [Pit13, §6.3] type theory, as well as combinations of these [BBC+19, CH20, RS17, WL20].[1] If we want to reap the fruit of such models *within* type theory and be able to write proofs that are unsound in simpler models, then we need internal operators reflecting some aspects of the model. While the constructions of presheaf models for various applications largely follow a common pattern, approaches towards internalization do not. Throughout the literature, various internal presheaf operators (the amazing right adjoint $\surd$ [LOPS18], Φ/extent and Ψ/Gel [BCM15, Mou16, CH20], Glue and Weld [CCHM17, NVD17], mill [ND18b], the strictness axiom [OP18] and locally fresh names [PMD15]) can be found with little or no analysis of their relative expressiveness. Understanding the common foundations of these operators would provide guidance to developers of new type systems, and may allow for cross fertilization between existing systems. For example, understanding that the Gel-type for parametricity – which so far has only been formulated w.r.t. an affine interval – is closely related to $\surd$ – which has been formulated w.r.t. the cartesian interval of cubical HoTT – can help us to generalize both to other flavours of intervals.

Three years ago [ND19], we proposed the *transpension* type $\between u : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma, u : \mathbb{U})$, right adjoint to structural or substructural universal quantification $\forall(u : \mathbb{U}) : \mathrm{Ty}(\Gamma, u : \mathbb{U}) \to \mathrm{Ty}(\Gamma)$ over a shape $\mathbb{U}$ (such as the interval), as a means of internalizing the peculiarities of presheaf models in general. Each of the aforementioned internal operators can be implemented from transpension, strictness and/or a pushout type former.[2] The transpension type has a structure reminiscent of a dependent version of the suspension type in HoTT [Uni13, §6.5]. In topoi, a right adjoint to structural quantification $\Pi(u : \mathbb{U})$ has already been considered by Yetter [Yet87], who named it ∇ and proved that it is definable from the amazing right adjoint $\surd$.

The structural transpension coquantifier $\between u$ is part of a sequence of adjoints $\Sigma\, u \dashv \Omega\, u \dashv \Pi\, u \dashv \between u$, preceded by the Σ-type, weakening and the Π-type. Adjointness of the first three is provable from the structural rules of type theory, but it is not immediately clear how to add typing rules for a further adjoint. Birkedal et al. [BCM+20] explain how to add a single modality that has a left adjoint in the semantics. If we want to have two or more adjoint modalities internally, then we can use a multimodal type system such as MTT [GKNB21, GKNB20].

In a paper currently under review [ND21] we present an extensional type system extending MTT, which features the transpension type as a modality and is backed by a presheaf model. Each internal modality in MTT needs a semantic left adjoint, so we can only internalize $\Omega\, u \dashv \Pi\, u \dashv \between u$. A drawback which we accept (as a challenge for future work), is that $\Omega\, u$ and $\Pi\, u$ become modalities which are a bit more awkward to deal with than ordinary weakening and Π-types. Below, we explain the main ideas of our approach, without reiterating the benefits and applications of the transpension type [ND19].

Shapes and Multipliers. Transpension is right adjoint to universal quantification over a shape. Because we want to support both structural (cartesian) and substructural (e.g. affine) quantification, our definition of shape needs to be a bit more general than just 'a representable object' of the presheaf model $\mathrm{Psh}(\mathcal{W})$ which would be essentially, via the Yoneda-embedding, an object of $\mathcal{W}$. Instead, a shape $\mathbb{U}$ will be modelled by an arbitrary endofunctor denoted $\sqcup \ltimes U : \mathcal{W} \to \mathcal{W}$,[3] dubbed a *multiplier*. We write U for a chosen object isomorphic to $\top \ltimes U$ so that we can always project $\pi_2 : W \ltimes U \to U$.

---

[1] We omit models that are not explicitly structured as presheaf models [AHH18, LH11, Nor19].

[2] For locally fresh names, we only have a heuristic translation.

[3] In a technical report [Nuy21], we generalize beyond *endo*functors.

Examples of such shapes are: the interval $\mathbb{I}$ in affine and cartesian cubical models of HoTT and/or parametricity, the sort of names in nominal type theory [PMD15], the sort of clocks of a given finite longevity in guarded type theory [BM20], and the twisted prism functor [PK20] which we believe is important for directed type theory.

Because arbitrary endofunctors are a bit too general to obtain many useful results, we introduce some criteria to classify multipliers. The multiplier gives rise to a functor $\exists_U : \mathcal{W} \to \mathcal{W}/U : W \mapsto (W \ltimes U, \pi_2)$ to the slice category over U. We say that $\sqcup \ltimes U$ is

- *semicartesian* if it is copointed, i.e. there is a first projection $\pi_1 : W \ltimes U \to W$,
- *cartesian* if it is a cartesian product,
- *cancellative* if $\exists_U$ is faithful (equivalently if $\sqcup \ltimes U$ is),
- *affine* if $\exists_U$ is full (which rules out being cartesian unless $U \cong \top$),
- *connection-free* if $\exists_U$ is essentially surjective on slices (V, φ) such that $\varphi : V \to U$ is *dimensionally split*, which in most cases just means split epi,
- *quantifiable* if $\exists_U$ has a left adjoint $\exists_U : \mathcal{W}/U \to \mathcal{W}$ (i.e. if $\sqcup \ltimes U$ is a local right adjoint).

For the properties of the example multipliers, we refer to the paper [ND21].

Modes are Shape Contexts. Every MTT judgement $p \mid \Gamma \vdash J$ is stated at some mode p, and modalities $\mu : p \to q$ have a domain and a codomain mode. The introduction rule for the modal type looks like this:

$$\frac{p \mid \Gamma, \text{🔒}_\mu \vdash a : A \qquad \mu : p \to q}{q \mid \Gamma \vdash \mathsf{mod}_\mu\, a : \langle \mu \mid A \rangle}$$

Typically (but not necessarily) every mode p will be modelled by a presheaf category $[\![p]\!]$ and every modality $\mu : p \to q$ will be modelled by a DRA [BCM+20] $[\![\text{🔒}_\mu]\!] \dashv [\![\mu]\!] : [\![p]\!] \to [\![q]\!]$.

A complication is that the modalities that we need, bind or depend on a variable, a phenomenon which is not supported by MTT. We solve this by grouping shape variables such as $u : \mathbb{U}$ in a *shape context* which is not considered part of the type-theoretic context but instead serves as the *mode* of the judgement. Formally, we define a shape context as any presheaf over $\mathcal{W}$, but in practice shape contexts will be denoted $(u_1 : \mathbb{U}_1, \ldots, u_n : \mathbb{U}_n)$ and obtained by applying (the left Kan extensions of) the corresponding multipliers to the terminal presheaf. Judgements in shape context Ξ are then interpreted in presheaves over the category of elements $\mathcal{W}/\Xi$, i.e. dependent presheaves over Ξ.

Modalities. As modalities $\mu : \Xi_1 \to \Xi_2$, we take *all* DRAs from $\mathrm{Psh}(\mathcal{W}/\Xi_1)$ to $\mathrm{Psh}(\mathcal{W}/\Xi_2)$. Again, a few specific ones are of special interest:

Modalities for substitution. A shape substitution (presheaf morphism) $\sigma : \Xi_1 \to \Xi_2$ leads to a functor $\Sigma^{/\sigma} : \mathcal{W}/\Xi_1 \to \mathcal{W}/\Xi_2$ which, by left Kan extension, precomposition and right Kan extension, leads to a triple of adjoint functors $\Sigma^{\sigma|} \dashv \Omega^{\sigma|} \dashv \Pi^{\sigma|} : \mathrm{Psh}(\mathcal{W}/\Xi_1) \to \mathrm{Psh}(\mathcal{W}/\Xi_2)$, the latter two of which can be internalized as modalities $\Omega\, \sigma \dashv \Pi\, \sigma$. Of course $\Omega\, \sigma$ is the substitution modality. In case σ is really a weakening $\pi : (\Xi, u : \mathbb{U}) \to \Xi$ over a semicartesian shape $\mathbb{U}$, then we write $\Omega\, u \dashv \Pi(u : \mathbb{U})$ and these stand for weakening and the Π-type.

Modalities for (co)quantification. A quantifiable multiplier gives rise to functors $\exists_U^{/\Xi} \dashv \exists_U^{/\Xi} : \mathcal{W}/\Xi \to \mathcal{W}/(\Xi, u : \mathbb{U})$, whence by Kan extension and precomposition a quadruple of adjoint functors $\exists_U^{\Xi|} \dashv \exists_U^{\Xi|} \dashv \forall_U^{\Xi|} \dashv \between_U^{\Xi|} : \mathrm{Psh}(\mathcal{W}/\Xi) \to \mathrm{Psh}(\mathcal{W}/(\Xi, u : \mathbb{U}))$, the latter three of which can be internalized as $\exists\, u \dashv \forall(u : \mathbb{U}) \dashv \between\, u$ which stand for fresh weakening, substructural quantification and transpension.

Theorem. If a multiplier $\sqcup \ltimes U$ is:

1. semicartesian, then we get morphisms $\mathsf{spoil}_u : \exists\, u \Rightarrow \Omega\, u$ and (hence) $\mathsf{cospoil}_u : \Pi\, u \Rightarrow \forall\, u$,
2. cartesian, then spoil_u and (hence) $\mathsf{cospoil}_u$ are isomorphisms,
3. cancellative and affine, then $\forall\, u \circ \exists\, u \cong 1$ and (hence) $\forall\, u \circ \between\, u \cong 1$,
4. cancellative, affine and connection-free, then the transpension type admits a pattern-matching eliminator not unlike that of the suspension type; equivalently, Φ/extent is then sound.

Acknowledgements We want to thank Jean-Philippe Bernardy, Dan Licata and Andrea Vezzosi for inspiring discussions on the subject. Andreas Nuyts holds a Postdoctoral Fellowship from the Research Foundation - Flanders (FWO). This work was partially supported by a research project of the Research Foundation - Flanders (FWO).

References

[AGJ14] Robert Atkey, Neil Ghani, and Patricia Johann. A relationally parametric model of dependent type theory. In *Principles of Programming Languages*, 2014. doi:10.1145/2535838.2535852.

[AHH18] Carlo Angiuli, Kuen-Bang Hou (Favonia), and Robert Harper. Cartesian Cubical Computational Type Theory: Constructive Reasoning with Paths and Equalities. In Dan Ghica and Achim Jung, editors, *Computer Science Logic (CSL 2018)*, volume 119 of *LIPIcs*, pages 6:1–6:17, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: http://drops.dagstuhl.de/opus/volltexte/2018/9673, doi:10.4230/LIPIcs.CSL.2018.6.

[BBC+19] Lars Birkedal, Aleš Bizjak, Ranald Clouston, Hans Bugge Grathwohl, Bas Spitters, and Andrea Vezzosi. Guarded cubical type theory. *Journal of Automated Reasoning*, 63(2):211–253, 8 2019. doi:10.1007/s10817-018-9471-7.

[BCH14] Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. In *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26, pages 107–128, Dagstuhl, Germany, 2014. URL: http://drops.dagstuhl.de/opus/volltexte/2014/4628, doi:10.4230/LIPIcs.TYPES.2013.107.

[BCM15] Jean-Philippe Bernardy, Thierry Coquand, and Guilhem Moulin. A presheaf model of parametric type theory. *Electron. Notes in Theor. Comput. Sci.*, 319:67 – 82, 2015. doi:http://dx.doi.org/10.1016/j.entcs.2015.12.006.

[BCM+20] Lars Birkedal, Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, and Bas Spitters. Modal dependent type theory and dependent right adjoints. *Mathematical Structures in Computer Science*, 30(2):118–138, 2020. doi:10.1017/S0960129519000197.

[BM20] Ales Bizjak and Rasmus Ejlers Møgelberg. Denotational semantics for guarded dependent type theory. *Math. Struct. Comput. Sci.*, 30(4):342–378, 2020. doi:10.1017/S0960129520000080.

[CCHM17] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *FLAP*, 4(10):3127–3170, 2017. URL: http://www.cse.chalmers.se/~simonhu/papers/cubicaltt.pdf.

[CH20] Evan Cavallo and Robert Harper. Internal parametricity for cubical type theory. In *28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain*, pages 13:1–13:17, 2020. doi:10.4230/LIPIcs.CSL.2020.13.

[CMS20] Evan Cavallo, Anders Mörtberg, and Andrew W Swan. Unifying Cubical Models of Univalent Type Theory. In Maribel Fernández and Anca Muscholl, editors, *Computer Science Logic (CSL 2020)*, volume 152 of *LIPIcs*, pages 14:1–14:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: https://drops.dagstuhl.de/opus/volltexte/2020/11657, doi:10.4230/LIPIcs.CSL.2020.14.

[GKNB20] Daniel Gratzer, Alex Kavvos, Andreas Nuyts, and Lars Birkedal. Type theory à la mode. Pre-print, 2020. URL: https://anuyts.github.io/files/mtt-techreport.pdf.

[GKNB21] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. Multimodal Dependent Type Theory. *Logical Methods in Computer Science*, Volume 17, Issue 3, July 2021. URL: https://lmcs.episciences.org/7713, doi:10.46298/lmcs-17(3:11)2021.

[Hof97] Martin Hofmann. *Syntax and Semantics of Dependent Types*, chapter 4, pages 79–130. Cambridge University Press, 1997.

[HS97] Martin Hofmann and Thomas Streicher. Lifting grothendieck universes. Unpublished note, 1997. URL: https://www2.mathematik.tu-darmstadt.de/~streicher/NOTES/lift.pdf.

[Hub16] Simon Huber. *Cubical Interpretations of Type Theory*. PhD thesis, University of Gothenburg, Sweden, 2016. URL: http://www.cse.chalmers.se/~simonhu/misc/thesis.pdf.

[KLV12] Chris Kapulkin, Peter LeFanu Lumsdaine, and Vladimir Voevodsky. The simplicial model of univalent foundations. 2012. Preprint, http://arxiv.org/abs/1211.2851.

[LH11] Daniel R. Licata and Robert Harper. 2-dimensional directed type theory. *Electr. Notes Theor. Comput. Sci.*, 276:263–289, 2011. doi:10.1016/j.entcs.2011.09.026.

[LOPS18] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. Internal universes in models of homotopy type theory. In *3rd International Conference on Formal Structures for Computation and Deduction, FSCD 2018, July 9-12, 2018, Oxford, UK*, pages 22:1–22:17, 2018. doi:10.4230/LIPIcs.FSCD.2018.22.

[Mou16] Guilhem Moulin. *Internalizing Parametricity.* PhD thesis, Chalmers University of Technology, Sweden, 2016. URL: publications.lib.chalmers.se/records/fulltext/235758/235758.pdf.

[ND18a] Andreas Nuyts and Dominique Devriese. Degrees of relatedness: A unified framework for parametricity, irrelevance, ad hoc polymorphism, intersections, unions and algebra in dependent type theory. In *Logic in Computer Science (LICS) 2018, Oxford, UK, July 09-12, 2018*, pages 779–788, 2018. doi:10.1145/3209108.3209119.

[ND18b] Andreas Nuyts and Dominique Devriese. Internalizing Presheaf Semantics: Charting the Design Space. In *Workshop on Homotopy Type Theory / Univalent Foundations*, 2018. URL: https://hott-uf.github.io/2018/abstracts/HoTTUF18_paper_1.pdf.

[ND19] Andreas Nuyts and Dominique Devriese. Dependable atomicity in type theory. In *TYPES*, 2019. URL: https://lirias.kuleuven.be/retrieve/540872.

[ND21] Andreas Nuyts and Dominique Devriese. Transpension: The right adjoint to the pi-type. *CoRR*, abs/2008.08533, 2021. URL: https://arxiv.org/abs/2008.08533, arXiv:2008.08533.

[Nor19] Paige Randall North. Towards a directed homotopy type theory. *Proceedings of the Thirty-Fifth Conference on the Mathematical Foundations of Programming Semantics, MFPS 2019, London, UK, June 4-7, 2019*, pages 223–239, 2019. doi:10.1016/j.entcs.2019.09.012.

[Nuy21] Andreas Nuyts. The transpension type: Technical report. *CoRR*, abs/2008.08530, 2021. URL: https://arxiv.org/abs/2008.08530, arXiv:2008.08530.

[NVD17] Andreas Nuyts, Andrea Vezzosi, and Dominique Devriese. Parametric quantifiers for dependent type theory. *PACMPL*, 1(ICFP):32:1–32:29, 2017. URL: http://doi.acm.org/10.1145/3110276, doi:10.1145/3110276.

[OP18] Ian Orton and Andrew M. Pitts. Axioms for modelling cubical type theory in a topos. *Logical Methods in Computer Science*, 14(4), 2018. doi:10.23638/LMCS-14(4:23)2018.

[Ort18] Ian Orton. *Cubical Models of Homotopy Type Theory - An Internal Approach.* PhD thesis, University of Cambridge, 2018.

[Pit13] Andrew M Pitts. *Nominal sets: Names and symmetry in computer science*, volume 57. Cambridge University Press, 2013.

[PK20] Gun Pinyo and Nicolai Kraus. From Cubes to Twisted Cubes via Graph Morphisms in Type Theory. In Marc Bezem and Assia Mahboubi, editors, *25th International Conference on Types for Proofs and Programs (TYPES 2019)*, volume 175 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:18, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. URL: https://drops.dagstuhl.de/opus/volltexte/2020/13069, doi:10.4230/LIPIcs.TYPES.2019.5.

[PMD15] Andrew M. Pitts, Justus Matthiesen, and Jasper Derikx. A dependent type theory with abstractable names. *Electronic Notes in Theoretical Computer Science*, 312:19 – 50, 2015. Ninth Workshop on Logical and Semantic Frameworks, with Applications (LSFA 2014). URL: http://www.sciencedirect.com/science/article/pii/S1571066115000079, doi:https://doi.org/10.1016/j.entcs.2015.04.003.

[RS17] E. Riehl and M. Shulman. A type theory for synthetic ∞-categories. *ArXiv e-prints*, May 2017. arXiv:1705.07442.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* http://homotopytypetheory.org/book, IAS, 2013.

[WL20] Matthew Z. Weaver and Daniel R. Licata. A constructive model of directed univalence in bicubical sets. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020*, pages 915–928. ACM, 2020. doi:10.1145/3373718.3394794.

[Yet87] David Yetter. On right adjoints to exponential functors. *Journal of Pure and Applied Algebra*, 45(3):287–304, 1987. URL: https://www.sciencedirect.com/science/article/pii/0022404987900776, doi:https://doi.org/10.1016/0022-4049(87)90077-6.