From iterated parametricity to indexed semi-simplicial and semi-cubical sets: a formal construction

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Abstract

The talk will remind how iterated parametricity connects to augmented semi-simplicial sets (unary case) and semi-cubical sets (binary case) and discuss the alternative between the fibered and indexed representations of parametricity. A construction of iterated parametricity in indexed form has been fully formalised in Coq which the talk will discuss.

An *augmented semi-simplicial set* is a presheaf from the category of strictly increasing functions over finite (possibly empty) totally ordered sets, that is, relying on the characterisation of strictly increasing functions from the most atomic ones, a family of sets X_n with face maps $d_i^n : X_{n+1} \to X_n$ for all $i \leq n$ such that $d_i^n \circ d_{j+1}^{n+1} = d_j^n \circ d_i^{n+1}$ for $i \leq j$. Let us call this definition the *fibered* definition of augmented semi-simplicial sets and consider the alternative definition obtained by iteratively applying the equivalence between functions to a set and families of sets indexed over this set (a degenerate form of Grothendieck's construction), that is, type-theoretically, between $\Sigma B : \mathsf{hSet}.(B \to A)$ and $A \to \mathsf{hSet}$ for $A : \mathsf{hSet}$. We call *indexed* definition of augmented semi-simplicial sets the alternative definition given by a family:

$$
\begin{array}{lcl}
X_0 & : & \mathsf{hSet} \\
X_1 & : & X_0. \to \mathsf{hSet} \\
X_2 & : & \Pi a_0 : X_0.\ X_1 a_0 \to X_1 a_0 \to \mathsf{hSet} \\
X_3 & : & \Pi a_0 : X_0.\ \Pi a_1 b_1 c_1 : X_1 a_0.\ X_2 a_0 a_1 b_1 \to X_2 a_0 a_1 c_1 \to X_2 a_0 b_1 c_1 \to \mathsf{hSet} \\
\vdots & &
\end{array}
$$

where, up to isomorphism, each declaration takes the form $X_{n+1} : \mathsf{frame}^n(X_0, ..., X_n) \to \mathsf{hSet}$ for a well-chosen definition of frame^n.

A possible recursive definition of frame^n was given in [Her15] (see also [Voe12, CK21]) but an alternative definition inspired from iterated parametricity translation can be given too. Indeed, it is known that the iteration of Reynolds' binary parametricity translation yields an indexed definition of semi-cubical sets, as sketched in [HM20] and studied in a categorical setting by Moeneclaey [Moe21] (about the relation between cubical sets and iterated parametricity, see also e.g. [AK18, GJF+15, JS17, CH20]). Moeneclaey[1] also noticed that the indexed definition of augmented semi-simplicial sets actually corresponds to the iteration of the unary version of the parametricity translation. The current talk is about giving the construction from [HM20] in all details, supported by a full formalisation in Coq, covering both the unary (that is augmented semi-simplicial) and binary (that is semi-cubical) cases.

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The informal intuition behind the construction was given in [HM20] which will be reminded in the talk. Fixing a universe level l, we now shortly explain the formal construction. It goes by inductively defining n-truncated sets $X_l^{<n} : \mathsf{U}_{l+1}$:

$$\begin{array}{lcl} X_l^{<0} & \triangleq & \mathsf{unit} \\ X_l^{<n'+1} & \triangleq & \Sigma D : X_l^{<n'}. (\mathsf{frame}_l^n(D) \to \mathsf{hSet}_l) \end{array}$$

then take the coinductive closure of it: $X_l \triangleq X_l^{\geq 0}(\star)$ where $\star : \mathsf{unit}$ defines the unit type, and, coinductively, $X_l^{\geq n}(D) \triangleq \Sigma X : (\mathsf{frame}_l^n(D) \to \mathsf{hSet}_l). X_l^{\geq n+1}(D, X)$. Itself, $\mathsf{frame}_l^n(D) \triangleq \mathsf{frame}_l^{n,n}(D) : \mathsf{hSet}_l$ is defined inductively by layers:

$$\begin{array}{lclcl} \mathsf{frame}_l^{n,0} & \star & \triangleq & \mathsf{unit} \\ \mathsf{frame}_l^{n,p'+1} & D & \triangleq & \Sigma d : \mathsf{frame}_l^{n,p'}(D). \mathsf{layer}_l^{n,p'}(d) \end{array}$$

where, for N the arity of the translation (here $N = 1$ or $N = 2$), a layer is made of N filled frames:

$$\mathsf{layer}_l^{n,p}(d) \triangleq \Pi\varepsilon : [1, N]. \mathsf{filler}_l^{n-1,p}(\mathsf{restr}_{\mathsf{frame},l,\varepsilon,p}^{n,p}(d))$$

where, for $D : X_l^{<n}$ and $E : \mathsf{frame}_l^n(D) \to \mathsf{hSet}_l$, we have that $\mathsf{filler}_l^{n,p} : \mathsf{frame}_l^{n,p}(D) \to \mathsf{hSet}_l$ is itself defined by reverse induction from p to n by:

$$\begin{array}{lclcl} \mathsf{filler}_l^{n,p,[p=n]} & (d) & \triangleq & E(d) \\ \mathsf{filler}_l^{n,p,[p<n]} & (d) & \triangleq & \Sigma l : \mathsf{layer}_l^{n,p}(d). \mathsf{filler}_l^{n,p+1}(d, l) \end{array}$$

In the definition of layers, for $D : X_l^{<n}$, $E : \mathsf{frame}_l^n(D) \to \mathsf{hSet}_l$ and $d : \mathsf{frame}_l^{n,p}(D)$, a family of restriction operators following the inductive structure of frames and playing the role of q-ε-face for the indexed construction is used:

$$\begin{array}{l} \mathsf{restr}_{\mathsf{frame},l,\varepsilon,q}^{n,p,[p\leq q<n]} : \mathsf{frame}_l^{n,p}(D) \to \mathsf{frame}_l^{n-1,p}(D.1) \\ \mathsf{restr}_{\mathsf{layer},l,\varepsilon,q}^{n,p,[p<q<n]} : \mathsf{layer}_l^{n,p}(d) \to \mathsf{layer}_l^{n-1,p}(\mathsf{restr}_{\mathsf{frame},l,\varepsilon,q}^{n,p}(d)) \\ \mathsf{restr}_{\mathsf{filler},l,\varepsilon,q}^{n,p,[p\leq q<n]} : \mathsf{filler}_l^{n,p}(d) \to \mathsf{filler}_l^{n-1,p}(\mathsf{restr}_{\mathsf{frame},l,\varepsilon,q}^{n,p}(d)) \end{array}$$

The key case is

$$\mathsf{restr}_{\mathsf{filler},l,\varepsilon,q}^{n,p,[p=q]}(b, \_) \triangleq b\,\varepsilon$$

but it also requires for $\mathsf{restr}_{\mathsf{layer},l}$ a coherence condition showing

$$\mathsf{restr}_{\mathsf{frame},l,\varepsilon,q}^{n-1,p}(\mathsf{restr}_{\mathsf{frame},l,\omega,r}^{n,p}(d)) = \mathsf{restr}_{\mathsf{frame},l,\omega,r}^{n-1,p}(\mathsf{restr}_{\mathsf{frame},l,\varepsilon,q+1}^{n,p}(d))$$

for $r \leq q$. This is also proved by following the inductive structure of frames and it eventually holds because we reason in hSet.

Taking into account coherences, the construction at level n requires to know what has been built at level $n-1$, $n-2$ and $n-3$. Working by full well-founded induction turned out to be difficult, so, in the formalisation[2], we restrict ourselves to maintain only the needed last 3 levels at each stage. Note that the formalisation takes great benefit of Coq's strict Prop to equate all syntactically distinct proofs of a given inequality occurring in the development. It additionally uses a representation of inequality proofs “à la Yoneda” (i.e. $n \leq_{\mathcal{Y}} p \triangleq \Pi q. q \leq n \to q \leq p$) to take benefit of even more definitional equalities in the proof.

[2]https://github.com/artagnon/bonak

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