Deciding the cofibration logic of cartesian cubical type theories

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Abstract

The defining feature of Homotopy Type Theory is the univalence principle, which identifies witnesses of propositional type equality with type equivalences. But because the standard version of HoTT [6] lacks a computational interpretation of this principle, cubical type theories were developed to remedy this situation. In HoTT, witnesses of propositional type equality are viewed as paths in a topological space. In turn, cartesian cubical type theories realize these paths as functions out of a special interval type $\mathbb{I}$. It may be understood as a synthetic analogue of the real interval $[0, 1]$.

At minimum, $\mathbb{I}$ supports reasoning about endpoints 0 and 1 and arbitrary elements x_i. It may additionally support reasoning about the max (i.e., join), min (i.e., meet), and complements of such elements. With all of these operations, $\mathbb{I}$ may be characterized as a free DeMorgan algebra on the x_i. With just max and min, $\mathbb{I}$ may be characterized as a free distributive lattice on the x_i. (For more on the design space of $\mathbb{I}$, see [2])

In cubical type theories, the structure of $\mathbb{I}$ is used to specify and reason about select portions of paths. This facility is essential to the definition of a cartesian cubical type theory. Portions of paths are described using a fragment of intuitionistic first-order equational logic over the terms of $\mathbb{I}$. Formulas of this language are called cofibrations. For example, in the context of a single variable x, $(x = 0) \vee (x = 1)$ is a cofibration which would select just the endpoints of a one-dimensional path. In the context of two variables x and y, $(x = y)$ is a cofibration which would select the diagonal of a two-dimensional cubical path (i.e., a solid square). Constraints on which equations are allowed (typically described as a choice in *generating* cofibrations) also contribute to variety in the design space of cubical type theories.

Given the goal of providing a computational interpretation of HoTT, it is of course crucial that the cofibration logic be decidable. Moreover, in the pursuit of implementation of cubical type theories, it is no less important that the decision procedure be practical. It is known that deciding cofibration entailment is coNP-hard [4]. But if this problem is coNP-complete for a particular cubical type theory, we consider this to be very good news. The size of the problems typically handled by proof assistants would be handled quickly by modern SAT solvers. Our focus therefore is on the availability of efficient reductions to the tautology problem for Boolean DNF formulas.

The current project emerged from the search for a practical decision procedure for a cubical type theory with a rich structure on $\mathbb{I}$—that of a free distributive lattice, coupled with an unconstrained notion of equational cofibration. This cubical structure is used in a cubical variant [7] of a synthetic theory of $(\infty, 1)$-categories [5], and the work in this talk was motivated by the problem of deciding judgmental equality in an implementation of this directed type theory. Entailment in this logic represents one of the hardest problems which arise from the cofibration logics of cartesian cubical type theories.

We compare this situation with that of two other cubical type theories whose implementation in proof assistants is active and ongoing. For [3], the interval structure is rich—i.e.,

that of a free DeMorgan algebra—but the equations are restricted to those which relate an arbitrary term in $\mathbb{I}$ to an endpoint, 0 or 1. This allows for an efficient factoring of an equation in a cofibration formula into a term over cofibrations of the form $(x = 0)$ or $(x = 1)$, which are join-prime. From here, the naive algorithm is to convert the cofibrations being compared into join-normal form, which has exponential run-time, and to factor the problem into ones whose antecedents are join-prime and to decide each individually, which can be done efficiently. For [1], the equations are unconstrained but the interval structure is minimal, with the result that the equations are always of the form $(x = 0)$, $(x = 1)$, or $(x = y)$, which are already join-prime. Hence the naive algorithm just described applies here as well.

Returning to the cubical type theory whose interval $\mathbb{I}$ is a free distributive lattice and whose cofibration equations are unconstrained, the naive approach is unlikely to be practical. To factor an equation into join-prime elements, the two terms being equated are first normalized. So unlike for the two cubical type theories just mentioned, factoring an equation into a cofibration over join-prime equations has exponential run-time. Hence, the naive decision procedure has double-exponential run-time.

In this talk, we will present a method for efficiently reducing cofibration entailment problems to the problem of deciding Boolean tautology of DNF formulas. Recalling that normalization trivializes the latter problem, this implies a significant practical speed up for all the cubical type theories mentioned above. We will also show correctness with respect to cubical presheaf semantics.

References

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