Type Theories with Universe Level Judgments

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History and state of the art. The system of simple type theory, as introduced by Church [2], is elegant and forms the basis of several proof assistants. However, it has some unnatural limitations: it is not possible in this system to talk about an arbitrary type, or about an arbitrary structure. It is also not possible to form the collection of e.g. all groups, as needed in category theory. In order to address these limitations, Martin-Löf [10, 9] introduced a system with a type V of all types. A function $A \to V$ in this system can then be seen as a family of types over a given type A, and it is natural in such a system to refine the operations of simple type theory, exponential and cartesian product, to operations of dependent products and sums. After the discovery of Girard's paradox in [5], Martin-Löf [11] introduced a distinction between *small* and *large* types, similar to the distinction introduced in category theory between large and small sets, and the type V became the (large) type of small types. The name "universe" for such a type was chosen in analogy with the notion of universes introduced by Grothendieck to represent category theory in set theory.

Later, Martin-Löf [12] introduced a countable tower of universes $\mathsf{U}_0 : \mathsf{U}_1 : \mathsf{U}_2 : \cdots$. We refer to the indices $0, 1, 2, \ldots$ as *universe levels*.

Before the advent of univalent foundations, most type theorists expected only the first few universe levels to be relevant in practical formalisations. This included the expectation that it might be feasible for a user of type theory to explicitly assign universe levels to their types, simply adding updated versions of earlier definitions when they were needed at different levels. However, the number of copies of definitions does not only grow with the level, but also with the number of type arguments in the definition of a type former. (The latter growth can be exponential!)

To deal with this problem Huet [8] and Harper and Pollack [6] and, in Coq, Sozeau and Tabareau [14] introduced *universe polymorphism*. Their "implicit" approach to universe polymorphisms is, however, problematic w.r.t. modularity, as pointed out in [3, 13]: one can prove $A \to B$ in one file, and $B \to C$ in one other file, while $A \to C$ is not valid. In order to cope with this issue, J. Courant [3] suggested to have explicit level universes, with a sup operation (see also [7]). This approach is now followed in Agda and in Voevodsky's proposal [16].

With the advent of Voevodsky's univalent foundations, the need for universe polymorphism has only increased, see for example [16]. The *univalence axiom* states that for any two types X, Y the canonical map

$$\mathsf{idtoeq}_{X,Y} : (X = Y) \to (X \simeq Y)$$

is an equivalence. Formally, the univalence axiom is an axiom scheme which is added to Martin-Löf type theory. If we work in Martin-Löf type theory with a countable tower of universes, each type is a member of some universe U_n. Such a universe U_n is *univalent* provided for all $X, Y : \mathsf{U}_n$ the canonical map $\mathsf{idtoeq}_{X,Y}$ is an equivalence. Let UA_n be the type expressing the univalence of U_n, and let $\mathsf{ua}_n : \mathsf{UA}_n$ for $n = 0, 1, \ldots$ be a sequence of constants postulating the respective instances of the univalence axiom. We note that $X = Y : \mathsf{U}_{n+1}$ and $X \simeq Y : \mathsf{U}_n$ and hence UA_n is in U_{n+1}. If we have a type of levels, as in Agda [15] or Lean [4], we can express universe polymorphism as quantification over universe levels.

We remark that universes are more important in a predicative framework than in an impredicative one. Consider for example the formalisation of real numbers as Dedekind cuts, or domain elements as filters of formal neighbourhoods, which belong to U_1 since they are properties of elements in U_0. However, even in a system using an impredicative universe of propositions, such as the ones in [8, 4], there is a need for the use of definitions parametric in universe levels.

Our contribution. The goal of this work is to complement the proposals by Courant [3] and Voevodsky [16] by handling constraints on universe levels and having instantiation operations. We start by giving the rules for a basic version of dependent type theory with Π, Σ, N, and an identity type former Id. We then explain how to add an externally indexed countable sequence of universes $\mathsf{U}_n, \mathsf{T}_n$ à la Tarski with or without cumulativity rules.

We introduce then an internal notion of universe level and add two new judgment forms: l Level meaning that l is a universe level, and $l = m$ meaning that l and m are equal universe levels. Here level expressions are built up from level variables α using a successor operation l^+ and a join (supremum, maximum) operation $l \vee m$. We let judgments depend not only on a context of ordinary variables, but also on a list of level variables $\alpha_1, \ldots, \alpha_k$, giving rise to a theory with level polymorphism. Certain typing rules are conditional on judgments of the form $l = m$. This is a kind of ML-polymorphism since we only quantify over global level variables.

We then extend the above type theory with formation rules for level-indexed product types $[\alpha]A$ meaning "A is a type for all universe levels α". Furthermore, introduction and elimination rules for such types are given, as well as some new computational rules. An example that uses level-indexed products is the following type which expresses the theorem that univalence for universes of arbitrary level implies function extensionality for functions between universes of arbitrary levels.

$$([\alpha]\mathsf{IsUnivalent}\,\mathsf{U}_\alpha) \to [\beta][\gamma]\mathsf{FunExt}\,\mathsf{U}_\beta\,\mathsf{U}_\gamma$$

We also present (a variation of) Voevodsky's proposal [16] with level constraints, complementing his proposal with a way to instantiate universe polymorphic constants introduced with level variables and constraints. We shortly discuss the decision problems for sup-semilattices with successor that come with this approach. These problems can be solved in polynomial time, as shown in [1].

As an example, we can define in our system a constant

$$c \;:\equiv\; \langle \alpha\ \beta \rangle \lambda_{Y:\mathsf{U}_\beta} \mathsf{Id}\ \mathsf{U}_\beta\ Y\ (\Sigma_{X:\mathsf{U}_\alpha} X \to Y) \quad : \quad [\alpha\ \beta][\alpha < \beta]\mathsf{U}_\beta \to \mathsf{U}_{\beta^+},$$

since $\Sigma_{X:\mathsf{U}_\alpha} X \to Y$ has type U_β in the context

$$\alpha : \mathsf{Level},\ \beta : \mathsf{Level},\ \alpha < \beta,\ Y : \mathsf{U}_\beta.$$

We can instantiate this constant c with two levels l and m, and this will be of type

$$[l < m]\mathsf{U}_m \to \mathsf{U}_{m^+},$$

which only can be used if $l < m$ holds in the current context.

In the current system of Agda [15], the constraint $\alpha < \beta$ is represented indirectly by writing β in the form $\gamma \vee \alpha^+$ and c is defined as

$$c \;:\equiv\; \langle \alpha\ \gamma \rangle \lambda_{Y:\mathsf{U}_{\alpha^+ \vee \gamma}} \mathsf{Id}\ \mathsf{U}_{\alpha^+ \vee \gamma}\ Y\ (\Sigma_{X:\mathsf{U}_\alpha} X \to Y) \quad : \quad [\alpha\ \gamma]\mathsf{U}_{\alpha^+ \vee \gamma} \to \mathsf{U}_{\alpha^{++} \vee \gamma^+},$$

which arguably is less readable. Moreover, not all constraints that occur in practice can be expressed in this way.

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