Homotopy setoids and quotient completion

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Introduction. A *setoid* is a notion of *set* in constructive mathematics, originally introduced by Bishop in [2]. In Martin-Löf intuitionistic type theory [13] a setoid can be described as a pair (X, R) where X is a closed type and R is a dependent type of the form

$$x_1, x_2 : \mathrm{X} \vdash \mathrm{R}(x_1, x_2)$$

satisfying suitable reflexivity, symmetry and transitivity conditions. The importance of setoids in type theory has been deeply investigated by Hofmann in [8] who introduced the *setoid model* in order to obtain *extensional constructs* in *intensional* type theory. Mainly, setoids are used to provide the *quotient* construction in place of the *quotient types* which lead to the undecidability of the type check or the introduction of non-canonical elements.

In category theory, setoids provide an instance of the categorical construction called the *exact completion* which is briefly described as follows. Given a weakly left exact category (wlex) $\mathscr{C}$, the exact completion $\mathscr{C}_{ex}$ of $\mathscr{C}$ is an exact category in the sense of Barr [1]. The objects of $\mathscr{C}_{ex}$ are given by the *pseudo equivalence relations* i.e. pairs of arrows of $\mathscr{C}$

$$r_1, r_2 : R \to X$$

satisfying suitable reflexivity, symmetry and transitivity conditions. An arrow between two pseudo relations $r_1, r_2 : R \to X$ and $s_1, s_2 : S \to Y$ is the equivalence class of a pair of arrows $(f, \tilde{f})$, where $f : X \to Y$ and $\tilde{f} : R \to S$, such that $f \circ r_i = s_i \circ \tilde{f}$, for $i = 1, 2$. The equivalence relation is given by the notion of *half-homotopy*, see [4]. Closed types and functions up to *functional extensionality* form a category **ML** which has strict finite products and weak pullbacks when the type theory considered is intensional. Hence, **ML** is a wlex category. The category **Std** of setoids and functions compatible with the relations has been widely studied, and it turns out to be equivalent to the exact completion of the category **ML**.

In order to study the properties of the exact completion $\mathscr{C}_{ex}$, one can verify if the category $\mathscr{C}$ shares a weaker version of these properties. For instance, in [3, Theorem 3.3] and [5, Theorem 3.6] the authors characterize the categories whose exact completion leads to a *local cartesian closed* category (lcc). In [7, Proposition 2.1], the authors characterize the categories whose exact completion is an *extensive* category. These results apply to the category of setoids which was already known to a be an lcc pretopos. Hence, we can summarize the above discussion in the following well-known results.

Fact. *The category* ***Std*** *is an lcc pretopos and* $\boldsymbol{ML}_{ex} \cong$ ***Std***.

Different parts of the proof of the first part of this fact can be found in [14] and [6].

Homotopy setoids. We have considered a homotopical version of setoids in view of ideas from the homotopy type theory [15]. A *homotopy setoids* is a setoid (X, R) such that the base type X is an *h-set* and the equivalence relation R is an *h-proposition*. By definition, h-props and h-sets are types such that the following types are inhabited

$$\mathsf{is\text{-}prop}(R) := \prod_{x,y:R} \mathsf{Id}_R(x, y) \qquad \mathsf{is\text{-}set}(X) := \prod_{x,y:X} \mathsf{is\text{-}prop}(\mathsf{Id}_X(x, y)).$$

Intuitively, h-propositions are types that are empty or contractible and h-sets are types that are *discrete*. The category $\mathbf{Std}_0$ is the full subcategory of $\mathbf{Std}$ of h-setoids and functions preserving relations. Our main objective is to prove that $\mathbf{Std}_0$ shares properties similar to $\mathbf{Std}$.

Problem. *The category $\boldsymbol{Std}_0$ is not exact and hence it is not the exact completion of a suitable category.*

A possible solution to this problem is to study homotopy setoids in the more general context of *elementary doctrines*. We recall that doctrines were introduced by Lawvere [9] and provide a categorical tool to work with syntactic logical theories. A *primary* doctrine is a functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ form a category $\mathscr{C}$ with finite products to the category of *partially ordered sets (posets)* and monotone functions. For instance, we can consider the functor $F^{ML} : \mathbf{ML}^{op} \to \mathsf{Pos}$ which associate to each closed type X, the poset of the types depending on X up to *logical equivalence*, i.e. two types A(x) and B(x) are equivalent when the type

$$x : X \vdash (A(x) \Rightarrow B(x)) \times (B(x) \Rightarrow A(x))$$

is inhabited. The action of F^{ML} on arrows is given by substitution of terms. If we denote with $\mathbf{ML}_0$ the full subcategory of $\mathbf{ML}$ of h-sets, then we can similarly consider the functor $F^{ML_0} : \mathbf{ML}_0^{op} \to \mathsf{Pos}$ which sends an h-set X to the poset of the types depending on X which are h-propositions. Actually, the above functors are *elementary* doctrines. These structures were introduced by Maietti and Rosolini in [11] and they are suitable primary doctrines which can deal with the equality predicate. This is achieved requiring, for each object $X \in \mathscr{C}$, the existence of an element $\delta_X \in \mathsf{P}(X \times X)$ such that

1. $\top_X \leq \mathsf{P}_{\Delta_X}\delta_X$
2. $\mathsf{P}_{p_1}\alpha \wedge \delta_X \leq \mathsf{P}_{p_2}\alpha$, for every $\alpha \in \mathsf{P}(X)$
3. $\mathsf{P}_{\langle p_1,p_3\rangle}\delta_X \wedge \mathsf{P}_{\langle p_2,p_4\rangle}\delta_y \leq \delta_{X\times Y}$, for every pair of objects $X, Y \in \mathscr{C}$.

For the elementary doctrines F^{ML} and F^{ML_0} the role of the equality is played by the identity type Id_X.

If P is an elementary doctrine, it is possible to define P-eq. relations and the corresponding notion of well-behaved quotients. In [11, 10, 12], the authors provide a construction which associates to each elementary doctrine P an elementary doctrine $\overline{\mathsf{P}} : \overline{\mathscr{C}}^{op} \to \mathsf{Pos}$, called the *elementary quotient completion* of P, with well-behaved quotients in a suitable universal way. The base category $\overline{\mathscr{C}}$ has objects given by pairs (X, ρ) with ρ a P-eq. relation on X. The arrows of $\overline{\mathscr{C}}$ are given by those arrows of $\mathscr{C}$ which preserve the P-equivalence relations. By construction, $\overline{\mathscr{C}}$ has quotients of all $\overline{\mathsf{P}}$-equivalence relations.

Example. The following are examples of elementary quotient completion:

1. The exact completion of a category with finite products and weak pullbacks is an instance of the elementary quotient completion for the elementary doctrine of *weak subobjects.*
2. The category $\mathbf{Std}$ is equivalent to the base category $\overline{\mathbf{ML}}$ and the category $\mathbf{Std}_0$ is equivalent to the base category $\overline{\mathbf{ML}_0}$.

We have provided a version of the [3, Theorem 3.3] and [7, Proposition 2.1] in the context of elementary doctrines and elementary quotient completion which generalize the statements for categories with strict products and weak pullbacks. Hence, we could define *relative pretoposes* as the elementary doctrines $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ with well-behaved quotients such that the category $\mathscr{C}$ is extensive. In this case, we say that the category $\mathscr{C}$ is a pretopos relative to P. We applied the results to F^{ML_0} and we obtained the following property for h-setoids.

Theorem. *The category $\boldsymbol{Std}_0$ is a lcc pretopos relative to $\overline{F^{ML_0}}$.*

References

[1] Michael Barr. Exact categories. In *Exact categories and categories of sheaves*, pages 1–120. Springer, 1971.

[2] Errett Bishop. Foundations of constructive analysis. 1967.

[3] Aurelio Carboni and Giuseppe Rosolini. Locally cartesian closed exact completions. *Journal of Pure and Applied Algebra*, 154(1-3):103–116, 2000.

[4] Aurelio Carboni and Enrico M Vitale. Regular and exact completions. *Journal of pure and applied algebra*, 125(1-3):79–116, 1998.

[5] Jacopo Emmenegger. On the local cartesian closure of exact completions. *Journal of Pure and Applied Algebra*, page 106414, 2020.

[6] Jacopo Emmenegger and Erik Palmgren. Exact completion and constructive theories of sets. *The Journal of Symbolic Logic*, 85(2):563–584, 2020.

[7] Marino Gran and EM Vitale. On the exact completion of the homotopy category. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 39(4):287–297, 1998.

[8] Martin Hofmann. Extensional concepts in intensional type theory. 1995.

[9] F William Lawvere. Adjointness in foundations. *Dialectica*, pages 281–296, 1969.

[10] Maria Emilia Maietti and Giuseppe Rosolini. Elementary quotient completion. *Theory and applications of categories*, 27(17):445–463, 2013.

[11] Maria Emilia Maietti and Giuseppe Rosolini. Quotient completion for the foundation of constructive mathematics. *Logica Universalis*, 7(3):371–402, 2013.

[12] Maria Emilia Maietti and Giuseppe Rosolini. Unifying exact completions. *Applied Categorical Structures*, 23(1):43–52, 2015.

[13] Per Martin-Löf and Giovanni Sambin. *Intuitionistic type theory*, volume 9. Bibliopolis Naples, 1984.

[14] Ieke Moerdijk and Erik Palmgren. Wellfounded trees in categories. *Annals of Pure and Applied Logic*, 104(1-3):189–218, 2000.

[15] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.