Homotopy setoids and quotient completion

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Introduction. A setoid is a notion of set in constructive mathematics, originally introduced by Bishop in [2]. In Martin-Löf intuitionistic type theory [13] a setoid can be described as a pair $(X, R)$ where $X$ is a closed type and $R$ is a dependent type of the form

$$x_1, x_2 : X \vdash R(x_1, x_2)$$

satisfying suitable reflexivity, symmetry and transitivity conditions. The importance of setoids in type theory has been deeply investigated by Hofmann in [8] who introduced the setoid model in order to obtain extensional constructs in intensional type theory. Mainly, setoids are used to provide the quotient construction in place of the quotient types which lead to the undecidability of the type check or the introduction of non-canonical elements.

In category theory, setoids provide an instance of the categorical construction called the exact completion which is briefly described as follows. Given a weakly left exact category (wlex) $\mathcal{C}$, the exact completion $\mathcal{C}_{ex}$ of $\mathcal{C}$ is an exact category in the sense of Barr [1]. The objects of $\mathcal{C}_{ex}$ are given by the pseudo equivalence relations i.e. pairs of arrows of $\mathcal{C}$

$$r_1, r_2 : R \rightarrow X$$

satisfying suitable reflexivity, symmetry and transitivity conditions. An arrow between two pseudo relations $r_1, r_2 : R \rightarrow X$ and $s_1, s_2 : S \rightarrow Y$ is the equivalence class of a pair of arrows $(f, \tilde{f})$, where $f : X \rightarrow Y$ and $\tilde{f} : R \rightarrow S$, such that $f \circ r_i = s_i \circ \tilde{f}$, for $i = 1, 2$. The equivalence relation is given by the notion of half-homotopy, see [4]. Closed types and functions up to functional extensionality form a category $\mathbf{ML}$ which has strict finite products and weak pullbacks when the type theory considered is intensional. Hence, $\mathbf{ML}$ is a wlex category. The category $\mathbf{Std}$ of setoids and functions compatible with the relations has been widely studied, and it turns out to be equivalent to the exact completion of the category $\mathbf{ML}$.

In order to study the properties of the exact completion $\mathcal{C}_{ex}$, one can verify if the category $\mathcal{C}$ shares a weaker version of these properties. For instance, in [3, Theorem 3.3] and [5, Theorem 3.6] the authors characterize the categories whose exact completion leads to a local cartesian closed category (lcc). In [7, Proposition 2.1], the authors characterize the categories whose exact completion is an extensive category. These results apply to the category of setoids which was already known to be an lcc pretopos. Hence, we can summarize the above discussion in the following well-known results.

Fact. The category $\mathbf{Std}$ is an lcc pretopos and $\mathbf{ML}_{ex} \cong \mathbf{Std}$.

Different parts of the proof of the first part of this fact can be found in [14] and [6].

Homotopy setoids. We have considered a homotopical version of setoids in view of ideas from the homotopy type theory [15]. A homotopy setoid is a setoid $(X, R)$ such that the base type $X$ is an $h$-set and the equivalence relation $R$ is an $h$-proposition. By definition, $h$-propos and $h$-sets are types such that the following types are inhabited

$$\text{is-prop}(R) := \prod_{x,y : R} \text{Id}_R(x, y) \quad \text{is-set}(X) := \prod_{x,y : X} \text{is-prop}(\text{Id}_X(x, y)).$$
Intuitively, h-propositions are types that are empty or contractible and h-sets are types that are discrete. The category $\text{Std}_0$ is the full subcategory of $\text{Std}$ of h-setoids and functions preserving relations. Our main objective is to prove that $\text{Std}_0$ shares properties similar to $\text{Std}$.

**Problem.** The category $\text{Std}_0$ is not exact and hence it is not the exact completion of a suitable category.

A possible solution to this problem is to study homotopy setoids in the more general context of elementary doctrines. We recall that doctrines were introduced by Lawvere [9] and provide a categorical tool to work with syntactic logical theories. A primary doctrine is a functor $P : \mathcal{C}^{\text{op}} \rightarrow \text{Pos}$ form a category $\mathcal{C}$ with finite products to the category of partially ordered sets (posets) and monotone functions. For instance, we can consider the functor $F^{\text{ML}} : \text{ML}^{\text{op}} \rightarrow \text{Pos}$ which associate to each closed type $X$, the poset of the types depending on $X$ up to logical equivalence, i.e. two types $A(x)$ and $B(x)$ are equivalent when the type

$$x : X \vdash (A(x) \Rightarrow B(x)) \times (B(x) \Rightarrow A(x))$$

is inhabited. The action of $F^{\text{ML}}$ on arrows is given by substitution of terms. If we denote with $\text{ML}_0$ the full subcategory of $\text{ML}$ of h-sets, then we can similarly consider the functor $F^{\text{ML}_0} : \text{ML}_0^{\text{op}} \rightarrow \text{Pos}$ which sends an h-set $X$ to the poset of the types depending on $X$ which are h-propositions. Actually, the above functors are elementary doctrines. These structures were introduced by Maietti and Rosolini in [11] and they are suitable primary doctrines which can deal with the equality predicate. This is achieved requiring, for each object $X \in \mathcal{C}$, the existence of an element $\delta_X \in P(X \times X)$ such that

1. $\top_X \leq P_{\Delta_X} \delta_X$
2. $P_p \alpha \land \delta_X \leq P_p \alpha$, for every $\alpha \in P(X)$
3. $P_{(p_1,p_3)} \delta_X \land P_{(p_2,p_4)} \delta_y \leq \delta_{X \times Y}$, for every pair of objects $X, Y \in \mathcal{C}$.

For the elementary doctrines $F^{\text{ML}}$ and $F^{\text{ML}_0}$ the role of the equality is played by the identity type $\text{Id}_X$.

If $P$ is an elementary doctrine, it is possible to define $P$-eq. relations and the corresponding notion of well-behaved quotients. In [11, 10, 12], the authors provide a construction which associates to each elementary doctrine $P$ an elementary doctrine $\overline{P} : \overline{\mathcal{C}}^{\text{op}} \rightarrow \text{Pos}$, called the elementary quotient completion of $P$, with well-behaved quotients in a suitable universal way. The base category $\overline{\mathcal{C}}$ has objects given by pairs $(X, \rho)$ with $\rho$ a $P$-eq. relation on $X$. The arrows of $\overline{\mathcal{C}}$ are given by those arrows of $\mathcal{C}$ which preserve the $P$-equivalence relations. By construction, $\overline{\mathcal{C}}$ has quotients of all $P$-equivalence relations.

**Example.** The following are examples of elementary quotient completion:

1. The exact completion of a category with finite products and weak pullbacks is an instance of the elementary quotient completion for the elementary doctrine of weak subobjects.
2. The category $\text{Std}$ is equivalent to the base category $\overline{\text{ML}}$ and the category $\text{Std}_0$ is equivalent to the base category $\overline{\text{ML}_0}$.

We have provided a version of the [3, Theorem 3.3] and [7, Proposition 2.1] in the context of elementary doctrines and elementary quotient completion which generalize the statements for categories with strict products and weak pullbacks. Hence, we could define relative pretoposes as the elementary doctrines $P : \mathcal{C}^{\text{op}} \rightarrow \text{Pos}$ with well-behaved quotients such that the category $\mathcal{C}$ is extensive. In this case, we say that the category $\mathcal{C}$ is a pretopos relative to $P$. We applied the results to $F^{\text{ML}_0}$ and we obtained the following property for h-setoids.

**Theorem.** The category $\text{Std}_0$ is a lcc pretopos relative to $F^{\text{ML}_0}$.
References


