Higher-Order Universe Operators in Martin-Löf Type Theory with one Mahlo Universe

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Background. Martin-Löf type theory has the so-called *universe types*. To provide the type-theoretic formulation of the Mahlo property, Setzer extended Martin-Löf type theory by means of a large universe type: he introduced Martin-Löf type theory **MLM** with one *Mahlo universe*, and determined its proof-theoretic ordinal [8, 9]. Mahlo universes have a reflection property similar to the ones of weakly Mahlo cardinals and recursively Mahlo ordinals.

On the other hand, Rathjen, Griffor and Palmgren [7] extended Martin-Löf type theory in another way: they introduced a system **MLQ** of Martin-Löf type theory. The system **MLQ** has the two types M and Q: roughly speaking, Q is an inductively defined set of codes for operators which provides universes closed under the universe operators constructed previously, and M is a universe closed under operators in Q. Using Aczel's interpretation [1, 2, 3] of constructive set theory **CZF** in Martin-Löf type theory, Rathjen, Griffor and Palmgren showed that **CZF** with an axiom asserting the existence of inaccessible sets of all transfinite orders is interpretable in **MLQ**. Moreover, the inductive construction of M and Q in **MLQ** was generalised by Palmgren [4]. He introduced a family $\mathbf{ML}^n$ of systems of *higher-order universe operators*, and showed that **MLQ** is an instance of these systems.

In sum, two powerful extensions of Martin-Löf type theory were introduced so far to study the type-theoretic counterparts of large sets: the extension by means of a reflection property similar to Mahloness (e.g. **MLM**), and the extension by means of higher-order universe operators (e.g. **MLQ** and $\mathbf{ML}^n$). The comparison between these two extensions in terms of their proof-theoretic strength was already attempted in the literature (see, for example, [4, 6, 9]). However, a more direct examination of the relationship between them is desirable as well.

Aim and Approach. We investigate the relationship between Mahlo universes and higher-order universe operators. Specifically, we show that higher-order universe operators in **MLQ** can be simulated in **MLM**, and extend this simulation to more general cases in $\mathbf{ML}^n$.

Below we use the logical framework adopted in the proof assistant Agda. We also adopt the families-of-sets formulation of **MLM** in [5], since this enables one to see the connection between Mahlo universes and higher-order universe operators more easily. Informally, a Mahlo universe type $\mathrm{V} : \mathrm{Set}$ "reflects" any operator on families of sets in V: for any $f : (\sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{V}) \to (\sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{V})$, where $\mathrm{T} : \mathrm{V} \to \mathrm{Set}$ is the decoding function for V, there is a subuniverse U_f of V with the decoding function $\widehat{\mathrm{T}}_f : \mathrm{U}_f \to \mathrm{V}$ such that U_f is closed under f. That is, we have

$$\frac{\Gamma \vdash f : (\sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{V}) \to (\sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{V})}{\Gamma \vdash \widehat{\mathrm{U}}_f : \mathrm{V}}\ \widehat{\mathrm{U}}\text{-}I \qquad\qquad \mathrm{T}\, \widehat{\mathrm{U}}_f = \mathrm{U}_f$$

The closedness property of U_f can be explained as follows. Define $\mathrm{T}_f : \mathrm{U}_f \to \mathrm{Set}$ as $\mathrm{T}_f a := \mathrm{T}(\widehat{\mathrm{T}}_f a)$. Then, the closedness under f is expressed by the operator $\mathsf{Res}_f : (\sum_{(a:\mathrm{U}_f)} \mathrm{T}_f a \to \mathrm{U}_f) \to (\sum_{(a:\mathrm{U}_f)} \mathrm{T}_f a \to \mathrm{U}_f)$ in **MLM**. This operator has the computation rule saying that

$\iota_f(\mathsf{Res}_f(a,b)) = f\ (\iota_f(a,b))$ holds for any $(a,b) : \sum_{(x:\mathrm{U}_f)} \mathrm{T}_f x \to \mathrm{U}_f$, where the injection $\iota_f : (\sum_{(x:\mathrm{U}_f)} \mathrm{T}_f x \to \mathrm{U}_f) \to (\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V})$ is defined as $\iota_f(a,b) := (\widehat{\mathrm{T}}_f a, \lambda x.\widehat{\mathrm{T}}_f(b\, x))$. The operator Res_f is intended as the restriction of f to the subuniverse U_f of V, and the injection ι_f shows that Res_f is indeed the restriction of f to U_f.

This reflection property enables Mahlo universes to define various universe operators. Before higher-order universe operators, we consider three examples: a universe above a family of sets in V, a usual universe operator and a super universe. A universe U above arbitrary $a : \mathrm{V}$ and $b : \mathrm{T}a \to \mathrm{V}$ is obtained by defining $f_0 : (\sum_{(x:\mathrm{V})} \mathrm{T}x \to \mathrm{V}) \to (\sum_{(x:\mathrm{V})} \mathrm{T}x \to \mathrm{V})$ as $f_0 := \lambda c.(a,b)$, where c does not occur in a nor b freely. Let $\widehat{\mathrm{N}}_{0,f_0}$ (resp. $\widehat{\mathrm{N}}_0$) be the code in U_{f_0} (resp. V) for the empty type, and C_0 be the eliminator for the empty type.

$$\iota_{f_0}(\mathsf{Res}_{f_0}(\widehat{\mathrm{N}}_{0,f_0}, \lambda x.\mathsf{C}_0 x)) = f_0(\iota_{f_0}(\widehat{\mathrm{N}}_{0,f_0}, \lambda x.\mathsf{C}_0 x)) = (\lambda c.(a,b))(\widehat{\mathrm{N}}_0, \lambda x.\widehat{\mathrm{T}}_{f_0}(\mathsf{C}_0 x)) = (a,b).$$

Therefore, the left projection $\mathsf{p}_1(\mathsf{Res}_{f_0}(\widehat{\mathrm{N}}_{0,f_0}, \lambda x.\mathsf{C}_0 x))$ of type U_{f_0} is a code in U_{f_0} for a, and the right projection $\mathsf{p}_2(\mathsf{Res}_{f_0}(\widehat{\mathrm{N}}_{0,f_0}, \lambda x.\mathsf{C}_0 x))$ is a code for b.

We define a super universe by reflecting a usual universe operator. To see this, note that an operator f which is reflected by V may have some parameters. For instance, when we have

$$y : \textstyle\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V} \vdash \lambda c.y : (\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V}) \to (\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V}),$$

we first obtain U_f with $f := \lambda c.y$ by the $\widehat{\mathrm{U}}$-I rule above. Then, we have the universe operator $\lambda y.(\widehat{\mathrm{U}}_f, \widehat{\mathrm{T}}_f) : (\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V}) \to (\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V})$ mapping any family of sets in V to a universe above this family. By the $\widehat{\mathrm{U}}$-I rule again, we obtain $\widehat{\mathrm{U}}_g : \mathrm{V}$ with $g := \lambda y.(\widehat{\mathrm{U}}_f, \widehat{\mathrm{T}}_f)$. Since g is a universe operator and U_g is closed under g, the universe U_g is a super universe.

One can treat higher-order universe operators as universe operators of type $\mathcal{O}_n$ with $n > 1$ in the sense of [4, Definition 5.1], where $\mathcal{O}_0 := \mathrm{Set}$, $\mathcal{F}_n := \sum_{(A:\mathrm{Set})} A \to \mathcal{O}_n$ and $\mathcal{O}_{n+1} := \mathcal{F}_n \to \mathcal{F}_n$. Note that $\mathcal{O}_1$ is the type of "first-order" operators, which are mappings from families of sets to themselves. Since our aim is to construct higher-order universe operators by means of the Mahlo universe V, we consider higher-order universe operators as universe operators of type O_n with $n > 1$, where $\mathrm{O}_0 := \mathrm{V}$, $\mathrm{F}_n := \sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{O}_n$ and $\mathrm{O}_{n+1} := \mathrm{F}_n \to \mathrm{F}_n$. We call an operator in O_n an *operator of order n*. For instance, the operator g above is of type O_1.

Higher-order universe operators in **MLQ** (see [7, § 3.2]) are, in our setting, operators whose inputs are a family (a_1, b_1) of type F_1 and a family (a_0, b_0) of type F_0. Then, these higher-order operators return a universe $\mathrm{U}(a_1, b_1, a_0, b_0)$ which is closed under the operator $b_1 x$ for each $x : \mathrm{T}\, a_1$, and includes (the codes of) (a_0, b_0). It is straightforward to transform these operators into universe operators of order 2 in the sense above, as explained in [4, Example 5.4].

Our idea for constructing higher-order universe operators is to use parameters in reflecting operators similarly to the case of the universe operator g above. As we utilised a family of sets in V as a parameter y in the case of g, the simulation of higher-order universe operators of **MLQ** requires a family of operators of order 1 as a parameter. Consider the following variables: $x : \mathrm{F}_1$, $e : \mathrm{T}(\mathsf{p}_1 x)$, $y : \mathrm{F}_0$ and $z : \mathrm{N}_2$ with the boolean type N_2. We define the operators h_0, h_1, h of order 1 as $h_0 := \lambda c.y$, $h_1 := \mathsf{p}_2 x\, e$ and $h := \mathsf{C}_2\, z\, h_0\, h_1$, where C_2 is the eliminator for N_2. Then, we have a universe $\widehat{\mathrm{U}}_h$ by reflection, and the family below can be defined:

$$(\widetilde{\mathrm{U}}, \widetilde{\mathrm{T}}) := \Big(\widehat{\Sigma}\, (\mathsf{p}_1 x)\, (\lambda e.(\widehat{\Sigma}\, \widehat{\mathrm{N}}_2(\lambda z.\widehat{\mathrm{U}}_h))),\ \ \lambda v.\widehat{\mathrm{T}}_{h'}(\mathsf{p}_2(\mathsf{p}_2 v))\Big) : \mathrm{F}_0 \text{ with } h' = h[\mathsf{p}_1 v/e][\mathsf{p}_1(\mathsf{p}_2 v)/z].$$

This family and Res_h simulate a universe which is closed under $\mathsf{p}_2 x\, e$ for each $e : \mathrm{T}\,(\mathsf{p}_1 x)$ and includes the family y. Finally, we obtain the higher-order operator $\lambda x.(\widehat{\mathrm{N}}_1, \lambda w.\lambda y.(\widetilde{\mathrm{U}}, \widetilde{\mathrm{T}})) : \mathrm{O}_2$, which takes arbitrary $x : \mathrm{F}_1$ and returns a universe closed under the operators in x. Operators of greater order in $\mathbf{ML}^n$ can be simulated as well by using parameters of greater order.

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