Higher-Order Universe Operators in Martin-L"of Type Theory with one Mahlo Universe

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**Background.** Martin-L"of type theory has the so-called *universe types*. To provide the type-theoretic formulation of the Mahlo property, Setzer extended Martin-L"of type theory by means of a large universe type: he introduced Martin-L"of type theory MLM with one *Mahlo universe*, and determined its proof-theoretic ordinal [8, 9]. Mahlo universes have a reflection property similar to the ones of weakly Mahlo cardinals and recursively Mahlo ordinals.

On the other hand, Rathjen, Griffor and Palmgren [7] extended Martin-L"of type theory in another way: they introduced a system MLQ of Martin-L"of type theory. The system MLQ has the two types M and Q: roughly speaking, Q is an inductively defined set of codes for operators which provides universes closed under the universe operators constructed previously, and M is a universe closed under operators in Q. Using Aczel’s interpretation [1, 2, 3] of constructive set theory CZF in Martin-L"of type theory, Rathjen, Griffor and Palmgren showed that CZF with an axiom asserting the existence of inaccessible sets of all transfinite orders is interpretable in MLQ. Moreover, the inductive construction of M and Q in MLQ was generalised by Palmgren [4]. He introduced a family ML\(^n\) of systems of *higher-order universe operators*, and showed that MLQ is an instance of these systems.

In sum, two powerful extensions of Martin-L"of type theory were introduced so far to study the type-theoretic counterparts of large sets: the extension by means of a reflection property similar to Mahloness (e.g. MLM), and the extension by means of higher-order universe operators (e.g. MLQ and ML\(^n\)). The comparison between these two extensions in terms of their proof-theoretic strength was already attempted in the literature (see, for example, [4, 6, 9]). However, a more direct examination of the relationship between them is desirable as well.

**Aim and Approach.** We investigate the relationship between Mahlo universes and higher-order universe operators. Specifically, we show that higher-order universe operators in MLQ can be simulated in MLM, and extend this simulation to more general cases in ML\(^n\).

Below we use the logical framework adopted in the proof assistant Agda. We also adopt the families-of-sets formulation of MLM in [5], since this enables one to see the connection between Mahlo universes and higher-order universe operators more easily. Informally, a Mahlo universe type V : Set “reflects” any operator on families of sets in V: for any \(f : (\sum_{a : V} T a \to V) \to (\sum_{a : V} T a \to V)\), where T : V \to Set is the decoding function for V, there is a subuniverse \(U_f\) of V with the decoding function \(\hat{T}_f : U_f \to V\) such that \(U_f\) is closed under \(f\). That is, we have

\[
\Gamma \vdash f : (\sum_{a : V} T a \to V) \to (\sum_{a : V} T a \to V) \quad \Gamma \vdash \hat{U}_f : V 
\]

\(\hat{T}_f : \hat{U}_f = U_f\)

The closedness property of \(U_f\) can be explained as follows. Define \(T_f : U_f \to Set\) as \(T_f a := T(\hat{T}_f a)\). Then, the closedness under \(f\) is expressed by the operator \(\text{Res}_f : (\sum_{a : U_f} T_f a \to U_f) \to (\sum_{a : U_f} T_f a \to U_f)\) in MLM. This operator has the computation rule saying that
$\iota_f(\text{Res}_f(a,b)) = f(\iota_f(a,b))$ holds for any $(a,b) : \sum_{(x:U_f)} T_f x \to U_f$, where the injection $\iota_f : (\sum_{(x:U_f)} T_f x) \to U_f \to (\sum_{(x:V)} T x \to V)$ is defined as $\iota_f(a,b) := (\tilde{T}_f a, \lambda x. \tilde{T}_f(b x))$. The operator $\text{Res}_f$ is intended as the restriction of $f$ to the subuniverse $U_f$ of $V$, and the injection $\iota_f$ shows that $\text{Res}_f$ is indeed the restriction of $f$ to $U_f$.

This reflection property enables Mahlo universes to define various universe operators. Before higher-order universe operators, we consider three examples: a universe above a family of sets in $V$, a usual universe operator and a super universe. A universe $U$ above arbitrary $a : V$ and $b : T a \to V$ is obtained by defining $f_0 : (\sum_{(x:V)} T x \to V) \to (\sum_{(x:V)} T x \to V)$ as $f_0 := \lambda c.(a,b)$, where $c$ does not occur in $a$ nor $b$ freely. Let $\tilde{N}_0.f_0$ (resp. $\tilde{N}_0$) be the code in $U_{f_0}$ (resp. $V$) for the empty type, and $\tilde{C}$ be the eliminator for the empty type.

$\iota_{f_0}(\text{Res}_{f_0}(\tilde{N}_0.f_0, \lambda x. C_0 x)) = f_0(\iota_{f_0}(\tilde{N}_0.f_0, \lambda x. C_0 x)) = (\lambda c.(a,b))(\tilde{N}_0, \lambda x. \tilde{T}_{f_0}(C_0 x)) = (a,b)$.

Therefore, the left projection $p_1(\text{Res}_{f_0}(\tilde{N}_0.f_0, \lambda x. C_0 x))$ of type $U_{f_0}$ is a code in $U_{f_0}$ for $a$, and the right projection $p_2(\text{Res}_{f_0}(\tilde{N}_0.f_0, \lambda x. C_0 x))$ is a code for $b$.

We define a super universe by reflecting a usual universe operator. To see this, note that an operator $f$ which is reflected by $V$ may have some parameters. For instance, when we have $y : \sum_{(x:V)} T x \to V \vdash \lambda c.y : (\sum_{(x:V)} T x \to V) \to (\sum_{(x:V)} T x \to V)$, we first obtain $U_f$ with $f := \lambda c.y$ by the $\tilde{U}$-$I$ rule above. Then, we have the universe operator

$\lambda y.(\tilde{U}_f, \tilde{T}_f) : (\sum_{(x:V)} T x \to V) \to (\sum_{(x:V)} T x \to V)$

mapping any family of sets in $V$ to a universe above this family. By the $\tilde{U}$-$I$ rule again, we obtain $\tilde{U}_g : V$ with $g := \lambda y.(\tilde{U}_f, \tilde{T}_f)$. Since $g$ is a universe operator and $U_g$ is closed under $g$, the universe $U_g$ is a super universe.

One can treat higher-order universe operators as universe operators of type $O_n$ with $n > 1$ in the sense of [4, Definition 5.1], where $O_0 := \text{Set}$, $\mathcal{F}_n := \sum_{(A:\text{Set})} A \to O_n$ and $O_{n+1} := \mathcal{F}_n \to \mathcal{F}_n$. Note that $O_1$ is the type of “first-order” operators, which are mappings from families of sets to themselves. Since our aim is to construct higher-order universe operators by means of the Mahlo universe $V$, we consider higher-order universe operators as universe operators of type $O_n$ with $n > 1$, where $O_0 := V$, $\mathcal{F}_n := \sum_{(a:V)} T a \to O_n$ and $O_{n+1} := \mathcal{F}_n \to \mathcal{F}_n$. We call an operator in $O_n$ an operator of order $n$. For instance, the operator $g$ above is of type $O_1$.

Higher-order universe operators in MLQ (see [7, § 3.2]) are, in our setting, operators whose inputs are a family $(a_1, b_1)$ of type $V_1$ and a family $(a_0, b_0)$ of type $F_0$. Then, these higher-order operators return a universe $U(a_1, b_1, a_0, b_0)$ which is closed under the operator $b_2$ for each $x : T a_1$, and includes (the codes of) $(a_0, b_0)$. It is straightforward to transform these operators into universe operators of order 2 in the sense above, as explained in [4, Example 5.4].

Our idea for constructing higher-order universe operators is to use parameters in reflecting operators similarly to the case of the universe operator $g$ above. As we utilised a family of sets in $V$ as a parameter $y$ in the case of $g$, the simulation of higher-order universe operators in MLQ requires a family of the universe operators of order 1 as a parameter. Consider the following variables:

$x : F_1, e : T(p_1 x), y : F_0$ and $z : N_2$ with the boolean type $N_2$. We define the operators $h_0, h_1, h_2$ of order 1 as $h_0 := \lambda c.y$, $h_1 := p_2 x e$ and $h := C_2 z h_0 h_1$, where $C_2$ is the eliminator for $N_2$. Then, we have a universe $\tilde{U}_h$ by reflection, and the family below can be defined:

$\tilde{U}(\tilde{U}, \tilde{T}) := (\tilde{\Sigma}(p_1 x)(\lambda e.(\tilde{\Sigma}\tilde{N}_2(\lambda z. \tilde{U}_h))), \lambda v. \tilde{T}_h(p_2(p_2 v))) : F_0$ with $h' = h[p_1 v/e][p_1(p_2 v)/z]$.

This family and $\text{Res}_{h}$ simulate a universe which is closed under $p_2 x e$ for each $e : T(p_1 x)$ and includes the family $y$. Finally, we obtain the higher-order operator $\lambda c.(\tilde{N}_1, \lambda e. \lambda y.(U,T)) : O_2$, which takes arbitrary $x : F_1$ and returns a universe closed under the operators in $x$. Operators of greater order in MLQ can be simulated as well by using parameters of greater order.
References


