Extending truth table natural deduction to predicate logic

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We extend *truth table natural deduction* (the method of deriving natural deduction rules for a connective c from the truth table t_c of c, as it has been introduced in [1, 2]) to predicate logic. We have two variations on the rules, one is a complete natural deduction calculus for classical logic, while the other is a complete natural deduction calculus for constructive logic.

Monotonic and non-monotonic quantifiers A main observation from earlier work is that for monotonic connectives (that is, connectives that come from a truth table that corresponds to a monotonic function from $\{0, \ldots, 1\}^n$ to $\{0, 1\}$, as induced by the $0 \leq 1$ ordering), the constructive and classical rules are equivalent [4]. For non-monotonic connectives: one classical non-monotonic connective makes the whole calculus classical. Naosuke Matsuda and Kento Takagi [3] extend this result by adding $\forall$ and $\exists$. For this, the Kripke models are restricted to those in which the (non-empty) domain $D(w)$ of individuals at w is a constant (instead of monotonic) function of w.

In this paper, we observe that quantifiers can themselves be monotonic or non-monotonic, and that they have constructive and classical deduction rules, which are equivalent for the monotonic connectives, but different for non-monotonic ones. For standard predicate logic, the existential quantifier $\exists$ is monotonic, while the universal quantifier $\forall$ is non-monotonic. This can be observed when thinking about Kripke models. If $\exists x.\varphi$ holds in a world w, it automatically holds in all worlds $w' \geq w$, because the domain element $d \in D(w)$ for which φ holds is still available in w' (or put differently: the truth of $\exists x.\varphi$ in a world can be defined locally). On the contrary, to know whether $\forall x.\varphi$ holds in a world w, it doesn't suffice to know that φ holds in w for all $d \in D(w)$. We also have to know this for all $d' \in D(w')$ for all worlds $w' \geq w$ (or put differently: the validity of $\forall x.\varphi$ in a world can only be define by inspecting all higher worlds).

There are various examples that show the difference between classical and constructive predicate calculus. One of them is known as the Drinker's Principle: $\exists y.(Dy \to \forall x.Dx)$. This can be proven by using a case distinction on $\forall x.Dx$ or $\exists x.\neg Dx$, and if one wants to limit the language to the connectives available in the formula, by using classical rules for implication. But it can also be shown using the classical rules for $\forall$ (to be given below) and constructive rules for implication. Another example is $\forall x.(Ax \vee Ex) \vdash \forall x.Ax \vee \exists x.Ex$, which we prove using our classical $\forall$-rules. (Note that if one uses the standard rules for $\exists$, $\forall$ and $\vee$, one has to use classical negation as an auxiliary connective to prove this formula!)

We present our derivation rules in a condensed form, leaving out auxiliary assumptions and conclusions. The rules are instantiated to derive sequents of the form $\Gamma; A \vdash \varphi$, where Γ is a finite set of formulas, φ is a single formula and A is a set of constants. We only give the rules for $\forall$ and $\exists$, as the rules for propositional connectives are standard (and in our format they can be found in [1, 2]).

First-order calculus Let L be a first-order language. For each finite set A of new individual constants we extend L to language $L[A]$. A *first-order sequent* $\Gamma; A \vdash \varphi$ consists of such a set A, a finite set of closed formulas of $L[A]$, Γ and a closed formula φ of $L[A]$. We recursively extend

L with new individual constants $a = a_\psi$ related to closed formulas ψ of the form $\exists x.\varphi$ or $\forall x.\varphi$. Constants added before may occur in ψ. In a classical model with (non-empty) domain D, the interpretation $d \in D$ of the constant a related to such a formula ψ should be such that the closed formulas ψ and $\varphi[a/x]$ are both true or both false in that model.

Inference rules are now operations on first-order sequents. The sequents used in the specification are differences of first-order sequents: only the parts that differ are written. The start rule (or "axiom") is extended as follows:

$$\frac{}{\varphi; A \vdash \varphi}\,S$$

We propose the following inference rules for the quantifiers. Rules indicated with a * are classical; similarly to the propositional connectives, only the introduction rules are different classically. For $\exists$, the classical and constructive introduction rule are equivalent.

$$\frac{\vdash \exists x.\varphi \qquad ; a \vdash}{\vdash}\,E\exists \qquad \frac{\vdash \exists x.\varphi \qquad \varphi[a/x]; \vdash}{\vdash}\,E\exists$$

$$\frac{\vdash \varphi[t/x]}{\vdash \exists x.\varphi}\,I\exists \qquad \frac{\exists x.\varphi; \vdash \qquad \vdash \varphi[t/x]}{\vdash}\,I^*\exists$$

$$\frac{\vdash \forall x.\varphi}{\vdash \varphi[t/x]}\,E\forall \qquad \frac{\vdash \forall x.\varphi \qquad \varphi[t/x]; \vdash}{\vdash}\,E\forall$$

$$\frac{; a \vdash}{\vdash \forall x.\varphi}\,I\forall \qquad \frac{\vdash \varphi[a/x]}{\vdash \forall x.\varphi}\,I\forall$$

$$\frac{\forall x.\varphi; \vdash \qquad ; a \vdash}{\vdash}\,I^*\forall \qquad \frac{\forall x.\varphi; \vdash \qquad \vdash \varphi[a/x]}{\vdash}\,I^*\forall$$

For the left $E\exists$ rule (and similarly for the left $I\forall$ rules), when we instantiate this rule to a concrete sequent, the a should not occur anywhere else in the formulas, otherwise the sequents are not well-formed. So a concrete application of the left $E\exists$ rule looks like this:

$$\frac{\Gamma_1; A_1 \vdash \exists x.\varphi \qquad \Gamma_2; A_2, a \vdash \chi}{\Gamma_1, \Gamma_2; A_1, A_2 \vdash \chi}\,E\exists$$

where a does not occur in Γ_2, χ, otherwise $\Gamma_1, \Gamma_2; A_1, A_2 \vdash \chi$ is not a first-order sequent.

Note that application of a constructive $\forall$-introduction rule can be seen as composition: application of the corresponding classical introduction rule in which the first premise is the result of a start rule application of the form $\forall x.\varphi; A \vdash \forall x.\varphi$.

We give one example of a natural deduction of a classically valid sequent which has the subformula property. We derive $\varphi; \vdash \chi$ where $\varphi = \forall x.(Ax \vee Ex)$ and $\chi = \forall x.Ax \vee \exists x.Ex$:

$$\dfrac{\dfrac{\dfrac{}{\forall x.Ax; \vdash \forall x.Ax}\,S}{\forall x.Ax; \vdash \chi}\,I\vee \qquad \dfrac{\dfrac{\dfrac{}{\varphi; a \vdash \forall x.(Ax \vee Ex)}\,S}{\varphi; a \vdash Aa \vee Ea}\,E\forall \quad \dfrac{\dfrac{\dfrac{}{Aa; a \vdash Aa}\,S}{Aa; a \vdash \forall x.Ax}\,I\forall}{Aa; a \vdash \chi}\,I\vee \quad \dfrac{\dfrac{\dfrac{}{Ea; a \vdash Ea}\,S}{Ea; a \vdash \exists x.Ex}\,I\exists}{Ea; a \vdash \chi}\,I\vee}{\varphi; a \vdash \chi}\,E\vee}{\varphi; \vdash \chi}\,I^*\forall$$

References

[1] H. Geuvers and T. Hurkens. Deriving natural deduction rules from truth tables. In *ICLA*, volume 10119 of *Lecture Notes in Computer Science*, pages 123–138. Springer, 2017.

[2] H. Geuvers and T. Hurkens. Proof Terms for Generalized Natural Deduction. In A. Abel, F. Nordvall Forsberg, and A. Kaposi, editors, *23rd International Conference on Types for Proofs and Programs (TYPES 2017)*, volume 104 of *LIPIcs*, pages 3:1–3:39, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[3] Naosuke Matsuda and Kento Takagi. Effect of the choice of connectives on the relation between the logic of constant domains and classical predicate logic. *CoRR*, arXiv:2107.03972, 2021. URL: https://doi.org/10.48550/arXiv.2107.03972, arXiv:2107.03972.

[4] I. van der Giessen. Natural deduction derived from truth tables, master thesis mathematics, radboud university nijmegen, July 2018. URL: https://www.ru.nl/math/@1060423/master-theses-per-year/.