Cubical Models are Cofreely Parametric

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Parametricity was originally introduced as a syntactic property [5], showing that terms of system F treat type input uniformly. It can be extended to the syntax of type theory [2]. We adopt a semantical point of view, so we define a parametric model of type theory as a model where:

- Any type comes with a chosen relation.
- Any term preserves these relations.
- This structure obeys equations, defining inductively parametricty on type and term constructors.

So being parametric is an additional structure on a model, witnessing some kind of uniformity for its terms. The usual syntactic parametricity can be recast by saying that the initial model is parametric. Many variants of parametricity have been studied, for example:

- Realizability, where every type comes with a predicate rather than a relation.
- Internal parametricity, where every type comes with a *reflexive* relation.

In this work we use clans as models of type theory (although everything can be adapted to lex categories or even plain categories). We prove that the category of clans is symmetric monoidal closed. Then we define notions of parametricity as monoids in this category, i.e. as monoidal models. We unfold this definition:

Lemma 1. *A notion of parametricity consists of a clan with a monoidal product such that:*

- *The monoidal product commutes with finite limits in both variables.*
- *Given fibrations $i \twoheadrightarrow i'$ and $j \twoheadrightarrow j'$ we have an induced fibration:*

$$i \otimes j \twoheadrightarrow (i' \otimes j) \underset{i' \otimes j'}{\times} (i \otimes j') \tag{1}$$

Then we define a parametric model for a notion of parametricity $\mathcal{M}$ simply as an $\mathcal{M}$-module. We give the following result:

Proposition 2. *Assume $\mathcal{U}$ a symmetric monoidal closed category with $\mathcal{M}$ a monoid in $\mathcal{U}$. Then the forgetful functor from $\mathcal{M}$-modules to $\mathcal{U}$ has:*

- *A left adjoint sending $\mathcal{C}$ to:*

$$\mathcal{M} \otimes \mathcal{C} \tag{2}$$

with $\mathcal{M}$ acting through the canonical left action of $\mathcal{M}$ on itself.

- *A right adjoint sending $\mathcal{C}$ to:*

$$\mathcal{M} \multimap \mathcal{C} \tag{3}$$

with $\mathcal{M}$ acting through the canonical right action of $\mathcal{M}$ on itself.

As a corollary, we have the following for any notion of parametricity:

Theorem 1. *The forgetful functor from parametric models to arbitrary ones has left and right adjoints, and we have compact descriptions for them.*

This extends the situation from [4] to any variant of parametricity, at the cost of forgetting arrow types and universes.

Then we give many examples of notions of parametricity. We build the following using the aforementioned right adjoints, for several variants of cubical objects:

- Categories of cubical objects.
- Lex categories of truncated cubical objects.
- Clans of Reedy fibrant cubical objects.

As an example, consider $\square$ the Reedy category of cubes with faces and reflexivities only. We have the following:

Proposition 3. *There exists a monoidal clan $\widehat{\square}$ such that for any clan $\mathcal{C}$, the clan:*

$$\widehat{\square} \multimap \mathcal{C} \tag{4}$$

is equivalent to the clan of Reedy fibrant functors from $\square$ to $\mathcal{C}$.

These results mean that many cubical models are cofreely parametric, giving a solid theoretical grounding to the observation that (variant of) cubical structures arise naturally when working with (variant of) parametricity [1,3].

References

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