A Generalized Translation of Pure Type Systems

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Introduction. The class of pure type systems was introduced by Terlouw [11] and Berardi [4] (and further developed by Barendregt [1, 2]) as a natural generalization of the lambda cube; it contains the lambda cube as well as systems with richer sort structure and quantification. Formally, a pure type system (PTS) λS is specified by a set of sorts $\mathcal{S}$, a set of axioms $\mathcal{A}$ satisfying $\mathcal{A} \subset \mathcal{S} \times \mathcal{S}$ and a set of rules $\mathcal{R}$ satisfying $\mathcal{R} \subset \mathcal{S} \times \mathcal{S} \times \mathcal{S}$, and has the same derivation rules as those given for the lambda cube except in the case of axioms and Π-type formation, which are replaced respectively with the following:

$$\vdash_{\lambda S} s_1 : s_2 \qquad\qquad \frac{\Gamma \vdash_{\lambda S} A : s_1 \qquad \Gamma, x : A \vdash_{\lambda S} B : s_2}{\Gamma \vdash_{\lambda S} \Pi x^A B : s_3}$$

where $(s_1, s_2) \in \mathcal{A}$ and $(s_1, s_2, s_3) \in \mathcal{R}$.

The study of pure type systems can be viewed as the study of how sort structure affects the meta-theoretic properties of a type system, especially because of the minimal set of type formers (*e.g.*, there are no Σ-types by default). One such meta-theoretic property, arguably one of the most important, is normalization. A type system is *weakly normalizing* if every typable term has a normal form and is *strongly normalizing* if no typable term appears in an infinite reduction sequence. Girard [6] demonstrated that sort structure can have a nontrivial effect on the normalization behavior of a type system by showing that the PTS λU is not strongly normalizing. In particular, circularity in the sort structure of a pure type system (λU does not explicitly include an axiom of the form "*Type* is a *Type*") is not a necessary condition for non-normalization. This leaves open a fundamental question: what is the relationship between the sort structure and normalization?

The last few decades have seen numerous techniques for proving normalization of systems in the lambda cube and their extensions, and some of these techniques have been extended to the PTS setting (*e.g.*, Melliès and Werner [9] extend the notion of Λ-sets to pure type systems) but many have not. The purpose of this abstract is to outline the generalization of one such technique, which might be called *dependency eliminating translations*.

Contributions. One approach for proving strong normalization of a type system is to define a typability-preserving infinite-reduction-path-preserving translation from that system into a weaker system which is already known to be strongly normalizing. Harper et al. [7] define such a translation from λP to $\lambda_{\to}$ and Geuvers and Nederhof [5] extend that translation to one from λC to $\lambda\omega$. Both of these translations can be viewed as deleting the *dependent* rule in the corresponding system, *i.e.*, the rule $(*, \Box)$, which allows types to depend on terms. For sufficiently well-structured pure type systems, this notion of dependence can be generalized, as is done by Barthe et al. [3] for their definition of *generalized non-dependent* pure type systems. I extend these translations to pure type systems in a way that maintains the property that dependent rules are deleted.

Before stating the following theorem, a few definitions. A PTS is *n-tiered* if it is of the form

$$\begin{aligned}
\mathcal{S} &= \{s_i \mid i \in [n]\} \\
\mathcal{A} &= \{(s_i, s_{i+1}) \mid i \in [n-1]\} \\
\mathcal{R} &\subset \{(s, s', s') \mid (s, s') \in \mathcal{S} \times \mathcal{S}\}
\end{aligned}$$

A tiered PTS is (i,j)-*full* if its rules contain $\{(s_l, s_k, s_k) \mid l \leq i \text{ and } l \leq k \leq j\}$ and is *full* if it satisfies the following closure property: if $(s_i, s_j, s_j) \in \mathcal{R}_{\lambda S}$ then λS is (j,i)-full. For any tiered PTS λS, define its *non-dependent restriction*, denoted λS^*, to be the tiered system with rules $\{(s_i, s_j, s_j) \in \mathcal{R}_{\lambda S} \mid i \leq j\}$.

Theorem 1. *For any full tiered PTS λS, there are two functions $\tau : \mathsf{T} \to \mathsf{T}$ and $[\![-]\!] : \mathsf{T} \to \mathsf{T}$ on terms such that the following hold.*

1. *If $\Gamma \vdash_{\lambda S} A : B$ then there is a context Γ' such that $\Gamma' \vdash_{\lambda S^*} [\![A]\!] : \tau(B)$. That is, $[\![-]\!]$ preserves typability.*

2. *For any term A derivable in λS, if $A \twoheadrightarrow_\beta B$, then $[\![A]\!] \twoheadrightarrow_\beta^+ [\![B]\!]$. That is, $[\![-]\!]$ preserves infinite reduction paths.*

This implies the strong normalization of any full tiered PTS depends only on the strong normalization of its non-dependent restriction. Unfortunately, fullness is a very strong property. A system which is (i,j)-full where $i \geq 2$ and $j \geq 3$, for example, contains λU and is thus inconsistent. There is only a small class of systems on which the translation can be applied non-trivially, and every system in this class is a subsystems of Lou's extended calculus of constructions (ECC) [8], which is known to be strongly normalizing. The strongest of these full n-tiered system has the rules

$$\{(s_k, s_1, s_1) \mid k \in [n]\} \cup \{(s_1, s_k, s_k) \mid k \in [n]\} \cup \{(s_2, s_k, s_k) \mid k \in [n]\}$$

Together with a proof that the non-dependent restriction of this system is strongly normalizing, we get a modular proof that this system is strongly normalizing along the lines of the Geuvers-Nederhof result. In particular, this proof does not require a detour through quasi-normalization (as is done by Luo for ECC) and the system for which one ultimately has to prove strong normalization after translation (by, say, the Girard-Tait method) is simpler.

Discussion. The restriction to tiered systems is for convenience, and it turns out to be sufficient even if we want to consider more general classes along the lines of persistent, stratified systems (see [3]) because such systems can be viewed as disjoint unions of tiered systems, in the sense of [10]. However, the restriction of fullness is clearly quite limiting. The generalization is, I believe, a fairly faithful one, so it seems possible that a more sophisticated translation could push this idea further. Being able to handle even just one additional non-dependent rule could be advantageous. For example, I became interested in translations like this one because of their potential application to the Barendregt-Guevers-Klop conjecture, an open question which posits that weak normalization implies strong normalization for all pure types systems. Barthe et al. [3] prove the conjecture holds for a class of non-dependent pure types systems via a CPS-style translation. If the non-dependent restriction of a tiered PTS is captured by the conditions of their theorem then their result can be leveraged and extended by a very simple bootstrapping argument: since weak normalization of λS implies weak normalization of λS^*, if λS is weakly normalizing then, in fact, λS^* is strongly normalizing, which implies λS is as well.

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