Extending Cubical Agda with Internal Parametricity

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  1. State it with internal, "synthetic", isomorphisms $p : \mathrm{Path}\ A\ B$
     CTT primitives yield $\circ$, $()^{-1}$, id
  2. Other primitives (e.g. Glue) ensuring univalence $\mathrm{Path}_{\mathcal{U}}\ A\ B \simeq (A \simeq B)$
     Univalence entails the pcpl.

Respecting relations(1)

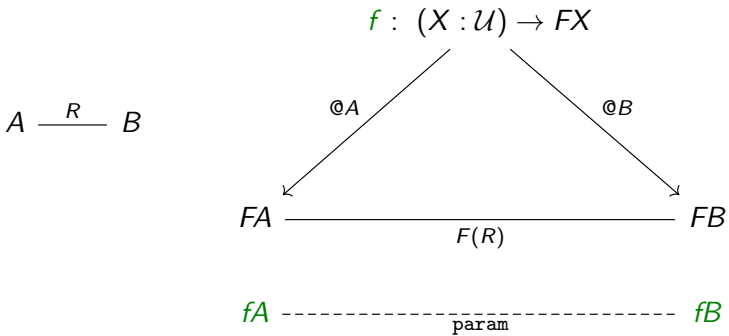
Param. TT. What about *relations*? 2 Observations.

1. Type-formers act on relations

Respecting relations(2)

Based on (1) we formulate parametricity:

2. *Polymorphic functions map type relations to provably related implementations*



The CH system, in Agda

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- Other primitives (extent, Gel) grant internal `param`
- We have implemented CH on top of `agda --cubical`
- We discuss extent, Gel, Bridge in `agda --bridges`
  $\rightsquigarrow$ `param`

Bridges: proofs of relatedness

```
  postulate
    BridgeP : ∀ {ℓ} (A : (x : BI) → Type ℓ) → (a0 : A bi0) → (a1 : A bi1) → Type ℓ
{-
  (A : (x : BI) → Type ℓ) relation ... BI-indexed type.
  BridgeP A a0 a1 = "a0 a1 are related under A"

      Γ , x:BI ⊢ q(x) : A(x)   +judg. endpoints
      ----------------------------------------------INTRO
      Γ ⊢ λ x . q(x) : BridgeP A a₀ a₁

      Γ ⊢ q : BridgeP A a0 a1   Γ ⊢ x : BI     Γ = Γ₁,x,Γ₂  FV(q) ⊆ Γ₁
      ----------------------------------------------ELIM
      Γ ⊢ q x : A x
-}

  refl-bridge : BridgeP (λ _ → A) a₀ a₀
  refl-bridge = λ x → a₀
```

extent: functions preserve bridges

Relational version of funext
Pointwise relatedness of f_0, f_1 implies relatedness

```
-- "related inputs map to related outputs"
Ptwise : (f₀ f₁ : A → B) → Type _
Ptwise f₀ f₁ = ∀ (a₀ a₁ : A) (aa : BridgeP (λ _ → A) a₀ a₁) → BridgeP (λ _ → B) (f₀ a₀) (f₁ a₁)

-- extent : Ptwise → Bridge
bridge-funext : (f₀ f₁ : A → B) → Ptwise f₀ f₁ ≃ BridgeP (λ _ → A → B) f₀ f₁
bridge-funext f₀ f₁ = isoToEquiv (iso
            (λ ptwise → λ x a → primExtent f₀ f₁ ptwise x a)
            (λ bdg a0 a1 aa x → bdg x (aa x))
            (λ bdg → {! ≈ extent-β !} )
            (λ ptwise → refl))

  -- Sometimes a=a'(x). Capture of a' ill defined if x not fresh.
  -- ⊢ primExtent f₀ f₁ ptwise x (a' x) = ptwise (a' bi0) (a' bi1) (λ j → a' j) x
```

Gel: bridges at Type are relations

```
-- Gel : (A₀ → A₁ → Type)  →  Bridge (λ _ → Type) A₀ B₀
relativity : (A₀ → A₁ → Type ℓ) ≃ BridgeP (λ x → Type ℓ) A₀ A₁
relativity = isoToEquiv (iso
  (λ R → λ x → primGel A₀ A₁ R x)
  (λ B → λ a0 a1 → BridgeP (λ x → B x) a0 a1 )
  (λ B →  {! not trivial. BridgeP at ≃ !} )
  (λ R →  {!...!} ))
```

Modular `param`

```
-- polym. functions map related types to related impl.
param-v0 : (F : Type ℓ → Type ℓ) (f : (X : Type ℓ) → F X) →
  (R : BridgeP (λ _ → Type ℓ) A₀ A₁) →
  BridgeP (λ x → F (R x)) (f A₀) (f A₁)
param-v0 F f = λ R → λ x → f ( R x )
```

Each time we use `param-v0` at specific F with custom F_{rel}, we have to prove

$$\ldots \circ F_{\text{rel}} \circ \ldots \equiv (\lambda R\, \lambda x.\, F(Rx))$$

We package such boilerplate proofs in a notion of *native relator*

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We package such boilerplate proofs in a notion of *native relator*
$\Rightarrow$ to use internal parametricity:

- (F, F_{rel}) native relator (e.g. by composition)
- `param` applies to all native relators.

Examples

Our `param` theorem grants one-liner proofs for

- Church encodings $\forall(X : \mathit{Type}).X \to X \to X \quad \simeq \quad \texttt{Bool}$
- (Predicative) System F has a parametric model
- (Not yet) the circle encoding $\forall(X_* : \mathit{Type}_*).\,\Omega(X_*) \to X \quad \simeq \quad S^1$
- Much more?

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Thank you

https://github.com/antoinevanmuylder/bridgy-lib