Gödel's Theorem Without Tears[1]

Essential Incompleteness in Synthetic Computability

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TYPES 2022

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[1] Abstract title: "Strong, Synthetic, and Computational Proofs of Gödel's First Incompleteness Theorem"

Gödel's First Incompleteness Theorem

Theorem

Any effective, consistent, and sufficiently powerful formal logic is incomplete.

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machine-checked	à la O'Connor (2005), Paulson (2014), and many others

Approaches to Incompleteness

Gödel: assuming ω-consistency

Gödel-Rosser approach

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early: assuming soundness

Kleene's approach

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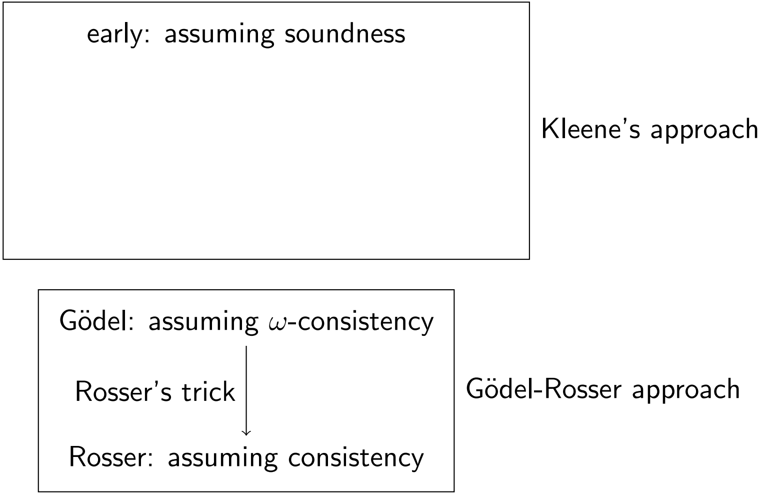
Approaches to Incompleteness



Approaches to Incompleteness

early: assuming soundness

↓ rec. insep. predicates

strengthened: assuming consistency[2]

Kleene's approach

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↓ Rosser's trick

Rosser: assuming consistency

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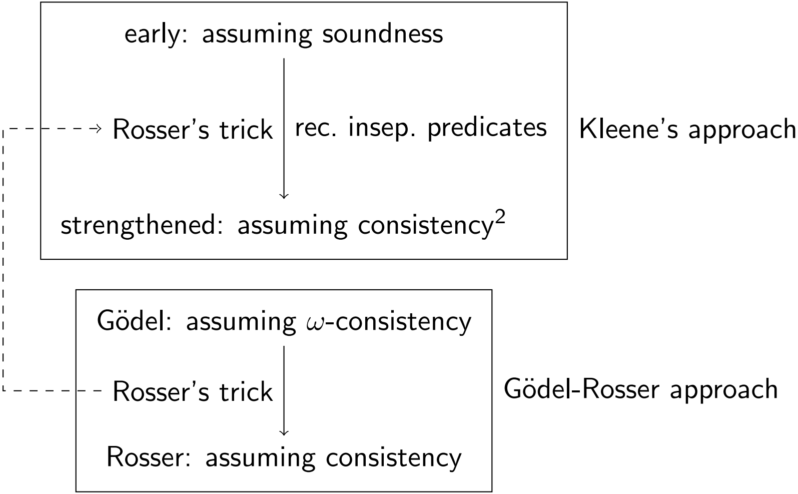

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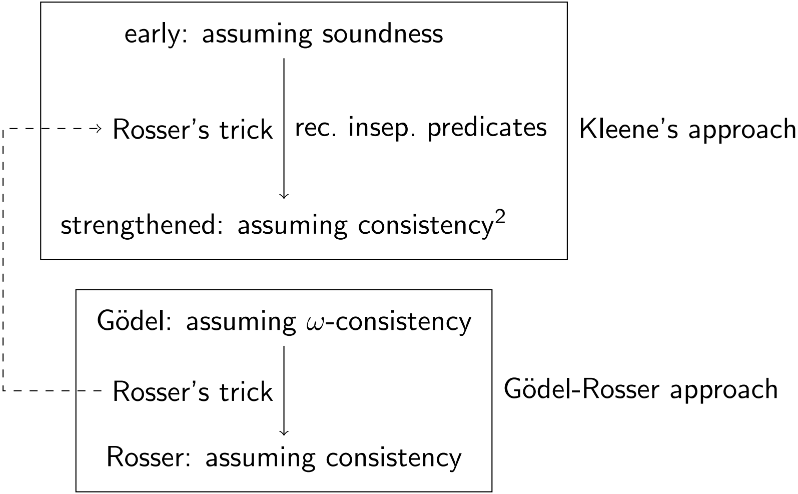

[2] We found out about these results through an e-mail by Anatoly Vorobey on the Foundations of Mathematics mailing list.

We factorised Kleene's incompleteness proofs into two parts:

1. Concise abstract core using synthetic computability
2. Instantiation of these abstract proofs to first-order logic using Rosser's trick

Abstract incompleteness proofs

- Kleene's early incompleteness result
- Improving Kleene's early result
- Kleene's strengthened incompleteness result

Instantiation to first-order Robinson arithmetic

Synthetic Computability[3]

We work in CIC, where all functions can be considered computable.

[3] Richman 1983; Bauer 2006.

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Definition

A predicate $P : X \to \mathbb{P}\mathrm{rop}$ is

- decidable if $\exists f : X \to \mathbb{B}.\, Px \leftrightarrow fx = \mathsf{true}$.

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Definition

A predicate $P : X \to \mathbb{P}\mathrm{rop}$ is

- decidable if $\exists f : X \to \mathbb{B}.\, Px \leftrightarrow fx = \mathsf{true}$.
- semi-decidable if $\exists f : X \to \mathbb{N} \to \mathbb{B}.\, \forall x.\, Px \leftrightarrow \exists k.\, fxk = \mathsf{true}$.

[3] Richman 1983; Bauer 2006.

Formal Systems

Definition (Formal system)

$\mathcal{F} = (S, \neg, \vdash)$ is a formal system if:

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$\mathcal{F}$ is complete if $\forall s.\, \mathcal{F} \vdash s \vee \mathcal{F} \vdash \neg s$.

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Many common formal logics are formal systems in this sense:

- first-order logic over a consistent and effective axiomatisation
- CIC
- ...

Decidable Formal Systems

Lemma

There is a partial function $d_{\mathcal{F}} : S \rightharpoonup \mathbb{B}$ separating provability from refutability:

$$\forall s.\, (d_{\mathcal{F}}\, s \triangleright \mathsf{true} \;\leftrightarrow\; \mathcal{F} \vdash s) \land (d_{\mathcal{F}}\, s \triangleright \mathsf{false} \;\leftrightarrow\; \mathcal{F} \vdash \neg s)$$

If $\mathcal{F}$ is complete, $d_{\mathcal{F}}$ is total.

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If $\mathcal{F}$ is complete, $d_{\mathcal{F}}$ is total.

Corollary

Any complete formal system is decidable.

Kleene's Early Incompleteness Proof[4,5]

Theorem

Let $\mathcal{F}$ be complete and weakly represent $P : \mathbb{N} \to \mathbb{P}\mathrm{rop}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.:

$$\forall x.\, Px \;\leftrightarrow\; \mathcal{F} \vdash rx$$

Then P is decidable.

[4] Kleene 1936; Turing 1936.
[5] As mechanised by Kirst and Hermes (2022).

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Then P is decidable. Thus, if P is undecidable, $\mathcal{F}$ is incomplete.

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Church's Thesis[7]

Axiom (EPF[6])

There is a function $\theta : \mathbb{N} \to \mathbb{N} \rightharpoonup \mathbb{B}$ such that:

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{B}.\, \exists c.\, f \equiv \theta c$$

[5]Kreisel 1967; Troelstra and van Dalen 1988.

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Definition (Self-halting problem)

The self-halting problem is defined as:

$$\mathcal{H} := \lambda x.\, \exists b.\, \theta x x \triangleright b$$

[5] Kreisel 1967; Troelstra and van Dalen 1988.
[6] Richman 1983; Forster 2022.

Self-halting problem

Fact

Partial functions $f : \mathbb{N} \rightharpoonup \mathbb{B}$ agreeing with the halting problem $\mathcal{H} := \lambda x.\, \exists b.\, \theta x x \triangleright b$:

$$\forall x.\, x \in \mathcal{H} \ \leftrightarrow \ f x \triangleright \text{true},$$

diverge on some input c, i.e., $\forall b.\, f c \not\triangleright b$.

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diverge on some input c, i.e., $\forall b.\, f c \not\triangleright b$.

Proof.

Consider $g : \mathbb{N} \rightharpoonup \mathbb{B}$,

$$g x := \begin{cases} \mathsf{false} & \text{if } f x \triangleright \mathsf{true} \\ \mathsf{undefined} & \text{otherwise.} \end{cases}$$

Let c be the code of g. We have $f c \triangleright \mathsf{true} \;\leftrightarrow\; f c \triangleright \mathsf{false}$. □

Strengthening the Early Incompleteness Proof[8]

Theorem

Assume $\mathcal{F}$ weakly represents $\mathcal{H}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.: $\forall x.\, x \in \mathcal{H} \leftrightarrow \mathcal{F} \vdash rx$
Then $\mathcal{F}$ has an independent sentence rc:

$$\mathcal{F} \nvdash rc \wedge \mathcal{F} \nvdash \neg rc$$

[8]Kleene 1952.

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Proof.

$d_{\mathcal{F}} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ agrees with the halting problem:

$$\forall x.\, d_{\mathcal{F}}\,(rx) \triangleright \mathsf{true} \;\leftrightarrow\; \mathcal{F} \vdash rx \;\leftrightarrow\; x \in \mathcal{H},$$

and therefore diverges on some input c. Thus, rc is independent in $\mathcal{F}$. □

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Going from Soundness to Consistency

- Consider weak representability:

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- Can we do better?

Definition (Strong Separability)

$\mathcal{F}$ strongly separates two predicates P_1, P_2 if there is an $r : \mathbb{N} \rightarrow S$ s.t.:

$$\forall x.\, P_1\, x \;\rightarrow\; \mathcal{F} \vdash rx \quad \wedge \quad P_2\, x \;\rightarrow\; \mathcal{F} \vdash \neg rx$$

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Recursively Inseparable Predicates

Theorem

Consider the following predicates:

$$\mathcal{I}_{\mathsf{true}} := \lambda x.\, \theta x x \triangleright \mathsf{true} \qquad \mathcal{I}_{\mathsf{false}} := \lambda x.\, \theta x x \triangleright \mathsf{false}$$

They are recursively inseparable, i.e., any partial function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ s.t.

$$\forall x.\, (x \in \mathcal{I}_{\mathsf{true}} \;\rightarrow\; f x \triangleright \mathsf{true}) \quad \wedge \quad (x \in \mathcal{I}_{\mathsf{false}} \;\rightarrow\; f x \triangleright \mathsf{false})$$

diverges on some input.

Kleene's Improved Incompleteness Proof[9]

Theorem

Assume $\mathcal{F}$ strongly separates $\mathcal{I}_{\text{true}}$ and $\mathcal{I}_{\text{false}}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.:

$$\forall x.\, x \in \mathcal{I}_{\text{true}} \;\to\; \mathcal{F} \vdash rx \quad \wedge \quad x \in \mathcal{I}_{\text{false}} \;\to\; \mathcal{F} \vdash \neg rx$$

$\mathcal{F}$ has an independent sentence rc:

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Proof.

$d_{\mathcal{F}} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ recursively separates $\mathcal{I}_{\text{true}}$ and $\mathcal{I}_{\text{false}}$, and therefore diverges on some input c. Therefore, rc is independent in $\mathcal{F}$. □

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Any (consistent) extension $\mathcal{F}'$ of $\mathcal{F}$ has an independent sentence rc:

$$\mathcal{F}' \nvdash rc \wedge \mathcal{F}' \nvdash \neg rc$$

Proof.

$d_{\mathcal{F}'} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ recursively separates $\mathcal{I}_{\text{true}}$ and $\mathcal{I}_{\text{false}}$, and therefore diverges on some input c. Therefore, rc is independent in $\mathcal{F}'$. □

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Lemma

$\mathrm{Q}' \subsetneq \mathrm{Q}$ weakly represents any semi-decidable predicate $P : \mathbb{N} \to \mathbb{P}\mathrm{rop}$ using a $\varphi \in \Sigma_1$:

$$\forall x.\, Px \;\leftrightarrow\; \mathrm{Q}' \vdash \varphi(\overline{x})$$

Proof.

See Kirst and Hermes (2022), relying on a mechanisation of the DPRM theorem by Larchey-Wendling and Forster (2022). □

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Goal: Show that Robinson arithmetic is strong enough to strongly separate any pair of semi-decidable and disjoint predicates.

Rosser's Trick for Strong Separability

Lemma (Strong Separability)

Q strongly separates any pair of semi-decidable and disjoint predicates P_1, P_2, i.e., there is some Φ s.t.:

$$\forall x.\, P_1\, x \;\rightarrow\; \mathrm{Q} \vdash \Phi(\overline{x}) \quad \wedge \quad P_2\, x \;\rightarrow\; \mathrm{Q} \vdash \neg\Phi(\overline{x})$$

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Proof.

Let φ_1, φ_2 be s.t. for any x:

$$P_1\, x \;\leftrightarrow\; \mathrm{Q} \vdash \exists k.\, \varphi_1(\overline{x}, k)$$
$$P_2\, x \;\leftrightarrow\; \mathrm{Q} \vdash \exists k.\, \varphi_2(\overline{x}, k)$$

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Choose:

$$\Phi(x) := \exists k.\, \varphi_1(x, k) \wedge \forall k' \leq k.\, \neg\varphi_2(x, k)$$

Instantiating the Strengthened Incompleteness Proof

Theorem

Robinson arithmetic is essentially incomplete.

$$\forall T \supseteq \mathrm{Q}. \quad T \text{ semi-decidable } \rightarrow \quad T \nvdash \bot \ \rightarrow \quad \exists \varphi. T \nvdash \varphi \land T \nvdash \neg\varphi$$

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Statement shown by Kirst and Hermes (2022):

$$\forall T \supseteq \mathrm{Q}.\, T \text{ semi-decidable } \to \mathbb{N} \vDash T \to (\forall \varphi.\, T \vdash \varphi \vee T \vdash \neg\varphi) \to \mathcal{H}_{\mathrm{TM}} \text{ decidable}$$

Summary

- Gave abstract incompleteness proofs due to Kleene in different strengths, reformulated and consolidated in synthetic computability
  - Assuming weak representability, using the halting problem
  - Assuming strong separability, using recursively inseparable predicates
  - Mechanised in only about 450 stand-alone lines of Coq, 200 for the strongest result

[10] Forster et al. 2020, notably including Larchey-Wendling and Forster 2022.
[11] Kirst, Hostert, et al. 2022.
[12] C.f. Hostert, Koch, and Kirst 2021.

Summary

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  - Mechanised in only about 450 stand-alone lines of Coq, 200 for the strongest result
- Instantiated those proofs to first-order Robinson arithmetic using Rosser's trick
  - Relying on libraries of undecidability[10] and first-order logic[11] and the first-order proofmode by Koch[12]
  - Mechanised in around 2200 lines of Coq
- Check our our development:

  `https://github.com/uds-psl/coq-synthetic-incompleteness/tree/types2022`

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Future Work

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- Do abstract proofs for a concrete model of computation
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- Gödel's second incompleteness theorem

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Church's Thesis

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{N}.\, \exists \varphi \in \Sigma_1.\, \forall xy.\, f x \triangleright y \;\leftrightarrow\; \mathrm{Q} \vdash \forall y'.\, \varphi(\overline{x}, y') \;\leftrightarrow\; y = y'$$

Rosser's Trick for Strong Separability

Let P_1, P_2 be semi-decidable and disjoint predicates, and $\varphi_1, \varphi_2 \in \Delta_0$ such that:

$$P_1\, x \;\leftrightarrow\; \mathrm{Q} \vdash \exists k.\, \varphi_1(\overline{x}, k) \qquad P_2\, x \;\leftrightarrow\; \mathrm{Q} \vdash \exists l.\, \varphi_2(\overline{x}, l)$$

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We want to find Φ_1 such that for all x:

$$P_1\, x \;\rightarrow\; \mathrm{Q} \vdash \exists k.\, \Phi_1(\overline{x}, k) \qquad P_2\, x \;\rightarrow\; \mathrm{Q} \vdash \neg\exists k.\, \Phi_1(\overline{x}, k)$$

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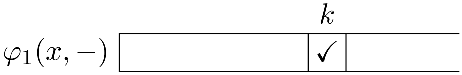

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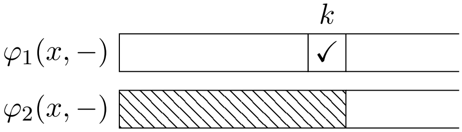

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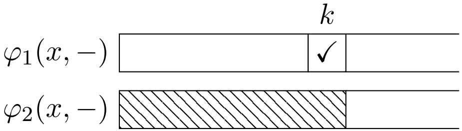

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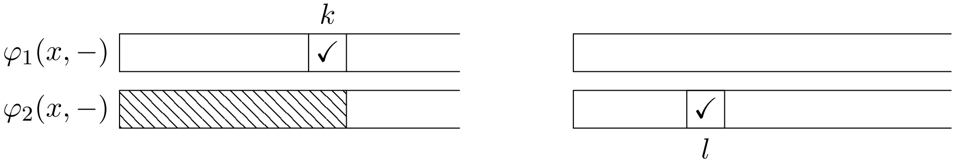

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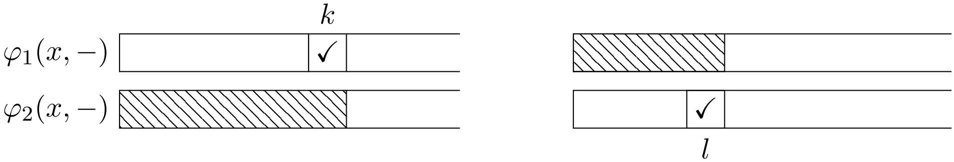

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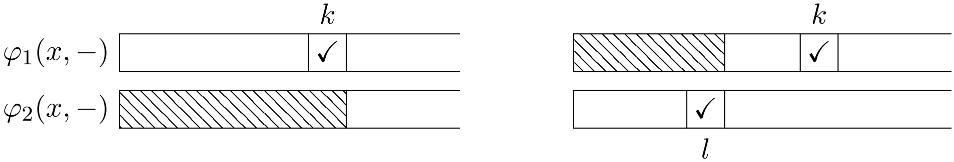

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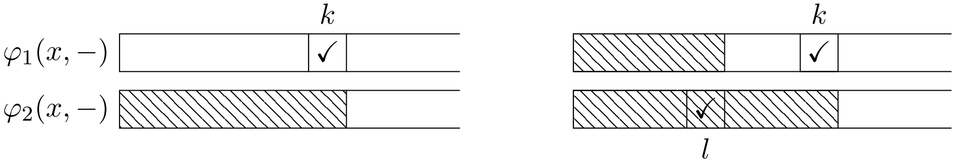

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