Semantics of two-dimensional type theory

Benedikt Ahrens
Paige Randall North
Niels van der Weide

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Higher dimensional type theory

- Martin-Löf’s identity type gives types the structure of higher groupoids
- This led to the development of homotopy type theory (HoTT)
- Synthetic algebraic topology: done via HoTT
- Directed type theory: directed version of HoTT
- Directed topological spaces are used to study concurrency \(^1\), and directed type theory is conjectured to model such spaces.

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There are many interpretations of type theory that are 2-dimensional in a certain sense

- The groupoid interpretation by Hofmann and Streicher \(^2\)
- The two-dimensional models by Garner \(^3\)

Interpreted in something like groupoids

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But directed variants have also been considered

- An interpretation with directed definitional equality\(^4\)
- A syntactical framework for directed type theory\(^5\)
- An interpretation with directed identity types\(^6\)

Interpreted in something like categories

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A framework is missing

**Problem:**
- Garner gave a general notion of 2-dimensional comprehension category, but this only works for **undirected** type theory
- The interpretations of directed type theory are ad hoc

**Goal of this talk:**

*find categorical framework in which one can interpret various flavors of 2-dimensional type theory*

The work in this talk has been formalized using UniMath.
Idea

- Use bicategories instead of categories
- Define **comprehension bicategories**.
- For that, we need a bicategorical notion of fibration\(^7\) \(^8\)
- Find suitable instances of comprehension bicategories

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Comprehension categories

Type theory can be interpreted in comprehesion categories.

Definition
A comprehesion category is a strictly commuting triangle

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\chi} & \mathcal{C} \\
\downarrow F & \ & \downarrow \mathrm{cod} \\
\mathcal{C} & \xleftarrow{\chi} & \mathcal{C}
\end{array}
\]

where $F$ is a Grothendieck fibration and where $\chi$ preserves cartesian cells.
Fibrations of bicategories

The notion of fibration of bicategories has a **global** and a **local** condition.
Fibrations of bicategories

The notion of fibration of bicategories has a \textbf{global} and a \textbf{local} condition.

\textbf{Global condition:}
Given a substitution $s : \Gamma_1 \to \Gamma_2$ and type $A$ in context $\Gamma_2$, we get a type $A[s]$ in context $\Gamma_1$. This is \textit{substitution on types}.
The notion of fibration of bicategories has a global and a local condition.

**Global condition:**
Given a substitution $s : \Gamma_1 \to \Gamma_2$ and type $A$ in context $\Gamma_2$, we get a type $A[s]$ in context $\Gamma_1$.
This is *substitution on types*.

**Local condition:**
Given a 2-cell $\tau : s_1 \Rightarrow s_2$ where $s_1, s_2 : \Gamma_1 \to \Gamma_2$, and a term $t : A[s_1]$, we get a term of type $A[s_2]$.
(think of 2-cells $\tau : s_1 \Rightarrow s_2$ as reductions from $s_1$ to $s_2$)
A **comprehension bicategory** is a *strictly* commuting triangle

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\chi} & \mathcal{B} \\
\downarrow F & & \downarrow \text{cod} \\
\mathcal{B} & & \\
\end{array}
$$

where $\chi$ preserves cartesian cells and where $F$ is a global fibration and a local opfibration.
Examples of comprehension bicategories

Given a **locally groupoidal** bicategory $\mathcal{B}$ with pullbacks, take

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{id} & \mathcal{B}\\
\downarrow{\text{cod}} & & \downarrow{\text{cod}}\\
\mathcal{B} & \xleftarrow{\text{cod}} & \mathcal{B}
\end{array}
$$

This does not work for arbitrary bicategories.
Examples of comprehension bicategories

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\end{array}
\]

This does **not** work for arbitrary bicategories.
Examples of comprehension bicategories

We have the following comprehension bicategory

\[ \text{OpFib} \longrightarrow \rightarrow \text{Cat} \]

\[ \text{cod} \longrightarrow \rightarrow \text{cod} \]

\[ \text{Cat} \]

This can be generalized to arbitrary bicategories by using \textit{internal Street (op)fibrations}.
Conclusion

- We defined a notion of **comprehension bicategory**
- This is a suitable framework in which one can interpret (directed) type theory: we proved **soundness**
- There are general instances of this definition (internal Street fibrations)
- More details can be found in the paper \(^9\).

Further work: look at type formers, completeness

\(^9\) Ahrens, Benedikt, North, Paige Randall, and Weide, Niels van der. "Semantics for two-dimensional type theory." *Accepted to LICS2022*