Patch Locale of a Spectral Locale in Univalent Type Theory

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23 June 2022
TYPES, Nantes, France
What is a locale?

Notion of space characterised solely by its frame of opens.
What is the patch locale?

What is a spectral locale?
A locale in which the compact opens form a basis closed under finite meets.

What is a Stone locale?
A compact locale in which the clopens form a basis.
What is the patch locale?

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Patch transforms **spectral locales** into **Stone ones**.
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Stone $\Rightarrow$ Spectral
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Patch transforms spectral locales into Stone ones. It is the universal such transformation.
Patch as a coreflector
Some examples of patch

**Spectral locale in consideration**

- Sierpiński space ($\Omega$)
- Scott topology of a (Scott) domain $\mathcal{P}(\mathbb{N}) \simeq \Omega^\mathbb{N}$
- Scott topology of domain $\mathbb{N}_\perp$

**Its patch**

- Booleans ($2$)
- Lawson topology
- Cantor space ($2^\mathbb{N}$)
- $\mathbb{N}_\infty$
Goal

Implement patch in univalent type theory *predicatively* i.e. without using resizing axioms.
Frames in type theory

Define \( \text{Fam}_\mathcal{W}(A) \equiv \Sigma_{i:\mathcal{W}} I \rightarrow A \).

Frame

A \((\mathcal{U}, \mathcal{V}, \mathcal{W})\)-frame consists of

- a type \( A : \mathcal{U} \),
- a partial order \(- \leq - : A \rightarrow A \rightarrow \text{hProp}_{\mathcal{V}}\),
- a top element \( \top : A \),
- a binary meet operation \(- \land - : A \rightarrow A \rightarrow A\),
- a join operation \( \bigvee _{-} : \text{Fam}_\mathcal{W}(A) \rightarrow A\),
- satisfying

\[
x \land \bigvee _{i:l} y_i = \bigvee _{i:l} x \land y_i
\]

for every \( x : A \) and family \( \{ y_i \}_{i:l} \) in \( A \).

The carrier type does not have to be explicitly required to be a set since this follows from the existence of a partial order on it.
Some notation

A frame homomorphism is a function preserving finite meets and arbitrary joins.

The category of frames and their homomorphisms is denoted $\text{Frm}$.

- The opposite category of $\text{Frm}$ is denoted $\text{Loc}$.
- Morphisms of $\text{Loc}$ are called continuous maps.

We pretend as though locales were spaces and use the letters
- $X$, $Y$, $Z$, \ldots for them;
- $f, g : X \to Y$ for their continuous maps; and
- $U, V : \mathcal{O}(X)$ for their opens.

The frame corresponding to a locale $X$ is denoted $\mathcal{O}(X)$ and the frame homomorphism corresponding to a continuous map $f : X \to Y$ is denoted $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$
Patch as the frame of Scott-continuous nuclei

A **nucleus on frame** $L$ is an endofunction $j : |L| \to |L|$ that is inflationary, idempotent, and preserves binary meets.

A nucleus is called **Scott-continuous** if it preserves joins of directed families.

**Patch of** $L$ is the **frame** formed by Scott-continuous nuclei on $L$. 
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This description of Patch was used by Escardó [1] to give a constructive, yet *impredicative*, treatment of the patch frame.
The frame of Scott-continuous nuclei in type theory?

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- **Our contribution:** we answer this question in the positive by constructing the frame of Scott-continuous nuclei in type theory *without using any resizing axioms*.
- This question turns out to be nontrivial.
Bases for frames

Consider a \((\mathcal{U}, \mathcal{V}, \mathcal{W})\)-locale \(X\).

**Defn. (Basis)**

A \(\mathcal{W}\)-family \(\{B_i\}_{i:\mathcal{I}}\) over a \((\mathcal{U}, \mathcal{V}, \mathcal{W})\)-locale \(X\) is said to **form a basis** for \(X\) if

for any \(U : \mathcal{O}(X)\), there is a **subfamily** \(\{B_i\}_{i:\mathcal{I}}\) of \(\{B_i\}_{i:\mathcal{I}}\) such that \(U = \bigvee_{i:\mathcal{I}} B_i\).
Bases for frames

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In our work, we are primarily interested in frames with bases of the form \((U^+, U, U)\) i.e.

large and locally small frames with small bases.
Spectrality revisited

Recall the impredicative definition of a spectral locale as one in which:

\[\text{the compact opens form a basis closed under finite meets.}\]

**Question:** How do we know that joins of covering families exist?
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- \( B_i \) is **compact** for each \( i : I \), and
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We use the same idea for Stone-ness.
Closed and open nuclei

Let $X$ be a spectral locale and $U : \mathcal{O}(X)$ an open.

We embed the opens of $X$ into $\text{Patch}(X)$ using the closed and open nuclei.

**Closed nucleus** of $U$: $\texttt{`U`} : \equiv V \mapsto U \lor V$

**Open nucleus** of $U$: $\neg \texttt{`U`} : \equiv V \mapsto U \Rightarrow V$
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Heyting implication
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Open nucleus of $U$: $\neg ‘U’ : \equiv V \mapsto U \supseteq V$.

**Problem:** it’s not so easy to write down the Heyting implication in the predicative context of type theory.

- The usual definition of Heyting implication (e.g. via the Adjoint Functor Theorem) is impredicative.
- We use (a version of the) Adjoint Functor Theorem for locally small frames with small bases.
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Formalised in modules $\text{AdjointFunctorTheoremForFrames}$, $\text{GaloisConnection}$, $\text{HeytingImplication}$ of Escardó’s $\text{TypeTopology}$ [0] Agda development.
Patch is Stone

Theorem

Given a spectral \((\mathcal{U}^+, \mathcal{U}, \mathcal{U})\)-locale \(X\) with a small basis \(\{B_i\}_{i:l}\), \(\text{Patch}(X)\) is a Stone locale.

Proof idea

The family

\[
\{\neg\neg 'B_k' \land 'B_i' \mid k, l : l\}
\]

forms a basis for \(\text{Patch}(X)\) and the covering subfamily for a given Scott-continuous nucleus \(j : \mathcal{O}(X) \to \mathcal{O}(X)\) is

\[
\{\neg 'B_k' \land 'B_i' \mid B_k \leq j(B_i), k, l : l\}
\]
We set out to implement a rather important construction of pointfree topology in univalent type theory, without using resizing.

Doing this predicatively turned out to involve surprising challenges.

We had to reformulate quite a few things in the theory itself to obtain a type-theoretic understanding of the construction in consideration.

Details can be found in our paper to appear at MFPS 2022.

Almost all of our work has been formalised in Agda, almost twice.
References I
