Patch Locale of a Spectral Locale in Univalent Type Theory

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What is a locale?

Notion of space characterised solely by its frame of opens.

What is the patch locale?

What is a spectral locale?

A locale in which the compact opens form a basis closed under finite meets.

What is a Stone locale?

A compact locale in which the clopens form a basis.

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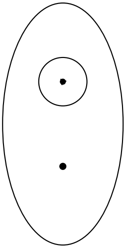
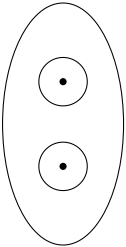
Patch transforms spectral locales into Stone ones.
It is the universal such transformation.

Patch as a coreflector

$$\mathbf{Stone} \underset{Patch}{\overset{i}{\underset{\bot}{\rightleftarrows}}} \mathbf{Spec}$$

Some examples of patch

Spectral locale in consideration	**Its patch**
Sierpiński space (Ω)	Booleans ($\mathbf{2}$)
	
Scott topology of a (Scott) domain	Lawson topology
$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$	Cantor space ($\mathbf{2}^{\mathbb{N}}$)
Scott topology of domain $\mathbb{N}_{\perp}$	$\mathbb{N}_{\infty}$

Goal

Implement patch in univalent type theory predicatively i.e. without using resizing axioms.

Frames in type theory

Define $\mathsf{Fam}_{\mathcal{W}}(A) :\equiv \Sigma_{I:\mathcal{W}} I \to A$.

Frame

A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-*frame* consists of

- a type $A : \mathcal{U}$,
- a partial order $- \leq - : A \to A \to \mathsf{hProp}_{\mathcal{V}}$,
- a top element $\top : A$,
- a binary meet operation $- \wedge - : A \to A \to A$,
- a join operation $\bigvee \_ : \mathsf{Fam}_{\mathcal{W}}(A) \to A$,
- satisfying
  $$x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$$
  for every $x : A$ and family $\{y_i\}_{i:I}$ in A.

The carrier type does not have to be explicitly required to be a set since this follows from the existence of a partial order on it.

Some notation

A frame homomorphism is a function preserving finite meets and arbitrary joins.

The category of frames and their homomorphisms is denoted **Frm**.

- The opposite category of **Frm** is denoted **Loc**.
- Morphisms of **Loc** are called continuous maps.

We pretend as though locales were spaces and use the letters

- $X, Y, Z, \ldots$ for them;
- $f, g : X \to Y$ for their continuous maps; and
- $U, V : \mathcal{O}(X)$ for their opens.

The frame corresponding to a locale X is denoted $\mathcal{O}(X)$ and the frame homomorphism corresponding to a continuous map $f : X \to Y$ is denoted $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$

Patch as the frame of Scott-continuous nuclei

A **nucleus** **on frame** L is an endofunction $j : |L| \to |L|$ that is inflationary, idempotent, and preserves binary meets.

A nucleus is called Scott-continuous if it preserves joins of directed families.

Patch **of** L is the frame formed by Scott-continuous nuclei on L.

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This description of Patch was used by Escardó [1] to give a constructive, yet *impredicative*, treatment of the patch frame.

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- This question turns out to be nontrivial.

Bases for frames

Consider a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locale X.

Defn. (Basis)

A $\mathcal{W}$-family $\{B_i\}_{i:I}$ over a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locale X is said to form a basis for X if

for any $U : \mathcal{O}(X)$, there is a *subfamily* $\{B_l\}_{l \in L}$ of $\{B_i\}_{i:I}$ such that $U = \bigvee_{l \in L} B_l$.

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In our work, we are primarily interested in frames with bases of the form $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$ i.e.

large and locally small frames with small bases.

Spectrality revisited

Recall the impredicative definition of a spectral locale as one in which:

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We use the same idea for Stone-ness.

Closed and open nuclei

Let X be a spectral locale and $U : \mathcal{O}(X)$ an open.

We embed the opens of X into $\mathsf{Patch}(X)$ using the closed and open nuclei.

Closed nucleus of U: $\quad$ ‘U’ $:\equiv V \mapsto U \vee V$.
Open nucleus of U: $\neg$‘U’ $:\equiv V \mapsto U \Rightarrow V$.

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Heyting implication

Problem: it's not so easy to write down the Heyting implication in the predicative context of type theory.

- The usual definition of Heyting implication (e.g. via the Adjoint Functor Theorem) is impredicative.
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Formalised in modules `AdjointFunctorTheoremForFrames`, `GaloisConnection`, `HeytingImplication` of Escardó's `TypeTopology` [0] Agda development.

Patch is Stone

Theorem

Given a spectral $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-locale X with a small basis $\{B_i\}_{i:I}$, Patch(X) is a Stone locale.

Proof idea

The family

$$\{`B_k\text{'} \wedge \neg `B_l\text{'} \mid k, l : I\}$$

forms a basis for Patch(X) and the covering subfamily for a given Scott-continuous nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ is

$$\{`B_k\text{'} \wedge \neg `B_l\text{'} \mid B_k \leq j(B_l), k, l : I\}$$

Summary

We set out to implement a rather important construction of pointfree topology in univalent type theory, without using resizing.

Doing this predicatively turned out to involve surprising challenges.

We had to reformulate quite a few things in the theory itself to obtain a **type-theoretic understanding** of the construction in consideration.

Details can be found in our paper to appear at MFPS 2022.

Almost all of our work has been formalised in *Agda*, almost twice.

References I

[1] Martín H. Escardó. "On the Compact-regular Coreflection of a Stably Compact Locale". In: *Electronic Notes in Theoretical Computer Science* 20 (1999), pp. 213–228. ISSN: 15710661. DOI: `10.1016/S1571-0661(04)80076-8`.

[0] Martín H. Escardó and contributors. *TypeTopology*. Agda library. URL: `https://github.com/martinescardo/TypeTopology` (visited on 01/22/2020).