# Linear lambda-calculus is linear

To appear at FSCD 2022

arXiv:2201.11221

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**28th International Conference on Types for Proofs and Programs (TYPES 2022)**

20-25 June 2022 – Nantes, France

# Motivation

**Linear Logic** $\longleftrightarrow$ **Linear Algebra**

Linearity expressed by its models (linear maps)

Lambda-calculus is the language of functions

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But, how?
Linear in the algebraic sense...

$$f(v+w) = f(v) + f(w)$$
$$f(a.v) = a.f(v)$$

This is not easily expressible with lambda-terms

# Interstitial rules

## Addition and multiplication by scalar

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A}{\Gamma \vdash A}\ \mathrm{sum} \qquad\qquad \frac{\Gamma \vdash A}{\Gamma \vdash A}\ \mathrm{prod}$$

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Now some proofs cannot be reduced, e.g.

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A} \quad \dfrac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge_i}{\Gamma \vdash A \wedge B}\ \textcolor{red}{\mathrm{prod}} \qquad \dfrac{\pi_3}{\Gamma, A \vdash C}}{\Gamma \vdash C}\ \wedge_e$$

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Commute sum either with the intro or with the elim

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A}}{\Gamma \vdash A}\ \textcolor{red}{\text{prod}} \quad \dfrac{\dfrac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash B}\ \textcolor{red}{\text{prod}}}{\Gamma \vdash A \wedge B}\ \wedge_i \qquad \dfrac{\pi_3}{\Gamma, A \vdash C}}{\Gamma \vdash C}\ \wedge_e \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A} \quad \dfrac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge_i \qquad \dfrac{\pi_3}{\Gamma, A \vdash C}}{\Gamma \vdash C}\ \wedge_e}{\Gamma \vdash C}\ \textcolor{red}{\text{prod}}$$

We chose to commute with the introduction as much as possible

# The $\mathcal{L}^{\mathcal{S}}$-calculus

## Intuitionistic linear logic connectives

| | Mult | Add | PL |
|---|---|---|---|
| Truth | $1$ | $\top$ | $\top$ |
| Falsehood | – | $0$ | $\bot$ |
| Implication | $\multimap$ | – | $\Rightarrow$ |
| Conjunction | $\otimes$ | $\&$ | $\wedge$ |
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## Proof-terms

$$
\begin{aligned}
t = x \mid t \boldsymbol{+} t \mid a \bullet t \mid a.\star \mid \delta_\top(t,t) \mid \delta_\bot(t) \\
\mid \lambda x.t \mid tt \\
\mid \langle t, t\rangle \mid \delta^1_\wedge(t, x.t) \mid \delta^2_\wedge(t, x.t) \\
\mid inl(t) \mid inr(t) \mid \delta_\vee(t, x.t, y.t)
\end{aligned}
$$

$a$ is scalar from some structure $(\mathcal{S}, +, \times)$.

## Some deduction rules

$$\frac{}{\vdash a.\star : \top}\ \top\text{-i}(a) \qquad \frac{\Gamma \vdash t : \top \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash \delta_\top(t, u) : A}\ \top\text{-e}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash t \boldsymbol{+} u : A}\ \text{sum} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash a \bullet t : A}\ \text{prod}(a)$$

## Some reduction rules

$$
\begin{aligned}
\delta_\top(a.\star, t) &\longrightarrow a \bullet t \\
a.\star \boldsymbol{+} b.\star &\longrightarrow (a + b).\star \\
(\lambda x.t) \boldsymbol{+} (\lambda x.u) &\longrightarrow \lambda x.(t \boldsymbol{+} u) \\
a \bullet b.\star &\longrightarrow (a \times b).\star \\
a \bullet \lambda x.t &\longrightarrow \lambda x.a \bullet t
\end{aligned}
$$

# Vectors and Matrices

**Definition (The set $\mathcal{V}$)**

$\top \in \mathcal{V}, \qquad \text{If } A, B \in \mathcal{V}, \quad A \wedge B \in \mathcal{V} \qquad\qquad A \in \mathcal{V}, \quad d(A) = \text{number of } \top$

**Definition (One-to-one correspondance)**

$$A \in \mathcal{V} \text{ with } d(A) = n \quad \longleftrightarrow \quad \mathcal{S}^n$$

$$\vdash t : A \ (t \text{ irreductible}) \longrightarrow \ \text{unique vector } \underline{t} \in \mathcal{S}^n$$

$$\mathbf{v} \in \mathcal{S}^n \longrightarrow \ \text{unique irreductible proof } \overline{\mathbf{v}}^A$$

**Example**

$$\underbrace{\langle\langle a.\star, b.\star\rangle, c.\star\rangle}_{(\top\wedge\top)\wedge\top} \leftrightarrow \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\mathcal{S}^3}$$

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**Lemma (Some properties)**

$$\underline{t \boldsymbol{+} u} = \underline{t} + \underline{u}$$

$$\underline{a \bullet t} = a\underline{t}$$

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**Theorem (Matrices)**

*$A, B \in \mathcal{V}$ with $d(A) = m$ and $d(B) = n$ $\quad M \in \mathcal{S}^{m\times n}$*
*Then, there exists $\vdash t : A \Rightarrow B$ such that for all $\mathbf{v} \in \mathcal{S}^m$*

$$\underline{t\overline{\mathbf{v}}^A} = M\mathbf{v}$$

**Example**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \lambda x.(\delta^1_\wedge(x, y.\delta_\top * (y, \langle a.\star, b.\star\rangle)) \boldsymbol{+} \delta^2_\wedge(x, z.\delta_\top * (\ \langle c.\star, d.\star\rangle)))$$

# Linearity

## The main theorem

**Theorem (Linearity)**

*Let $A$ be any proposition, $B \in \mathcal{V}$, $\vdash t : A \Rightarrow B$, and $u, v \in A$, then*

$$t(u \mathbin{\boldsymbol{+}} v) \equiv (tu) \mathbin{\boldsymbol{+}} (tv) \qquad t(a \bullet u) \equiv a \bullet (tu)$$

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$$t(u + v) \equiv (tu) + (tv) \qquad t(a \bullet u) \equiv a \bullet (tu)$$

It does not extend directly to arbitrary $B$:
Let $t = \lambda x.\lambda y.(yx) : \top \Rightarrow (\top \Rightarrow \top) \Rightarrow \top$

$$t(1.\star + 2.\star) \longrightarrow^* \lambda y.(y3.\star)$$
$$(t1.\star) + (t2.\star) \longrightarrow^* \lambda y.((y1.\star) + (y2.\star))$$

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**Corollary**

*Let $A, B \in \mathcal{V}$ with $d(A) = m$, $d(B) = n$ and $\vdash t : A \Rightarrow B$.
Then, the map $F$ from $\mathcal{S}^m$ to $S^n$, defined as $F(\mathbf{u}) = \underline{t\overline{u}^A}$ is linear.*

# Summarizing

- We presented the $\mathcal{L}^{\mathcal{S}}$-logic, an extension of a fragment of intuitionistic linear logic with two interstitial rules and its commuting rules to recover cut elimination.
- We have shown that its proof-language can express matrices in $\mathcal{S}^{m\times n}$
- Moreover, we have shown that every term of type $A \Rightarrow B$, with $B \in \mathcal{V}$ is linear in the algebraic sense.

## In the paper, but I got no more time for this...

- Extending the $\mathcal{L}^{\mathcal{S}}$-logic with the $\odot$ operator, we get a quantum programming language (see `[Díaz-Caro & Dowek ICTAC 2021]`).