Towards a Formalization of Affine Schemes in `Cubical Agda`

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- First constructive account due to Joyal [1976]
- Above definition of ZL due to Español [1983]
- Formalization generally follows Coquand et al. [2009]

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⇒ Formalize constructive approach in `Cubical Agda` (with crucial help from univalence)

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$\Rightarrow$ Great for constructive and univalent formalization
of set-level mathematics in the spirit of Voevodsky [2015]

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Also need (wip):

- Proof of sheaf property
- Extension to ZL preserves sheaf property

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Buuut: can *actually* employ univalence through a simple algebraic trick

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And by using univalence/the SIP:

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- What we *actually* need for constructing the presheaf
- Lets us avoid cumbersome diagram chases in proof of sheaf property

Summary

We presented the outline of a formalization of affine schemes that:

- Uses a synthetic description of Zariski lattice but follows (& elaborates) the textbook strategy $D(f) \mapsto R[1/f]$
- Uses a simple algebraic observation and univalence to make the construction work *out of the box*!

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We presented the outline of a formalization of affine schemes that:

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There is still some work to be done:

- Finish the construction of the structure sheaf (Extension of sheaves & sheaf property for general covers) Then sanity-check: $\mathcal{O}\big(D(1)\big) \equiv R$
- Future work: Define (spectral) schemes & projective schemes

Thank You

Sheaves

Presheaf $\mathcal{F} : L^{op} \to \mathcal{C}$ is *sheaf on distributive lattice* L if:

- $\mathcal{F}(\bot)$ is the terminal object in $\mathcal{C}$
- $\forall\, x, y \in L$ the following is a pullback square

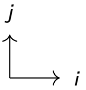
$$\begin{array}{ccc} \mathcal{F}(x \vee y) & \longrightarrow & \mathcal{F}(x) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(y) & \longrightarrow & \mathcal{F}(x \wedge y) \end{array}$$

Action of the structure sheaf on objects

$$\Sigma[\, \mathfrak{a} \in \mathrm{ZL} \,] \left(\underbrace{\exists[\, f \in R \,]\, (D(f) \equiv \mathfrak{a})}_{\text{h-prop}} \right) \to \underbrace{\text{R-Alg}}_{\text{h-groupoid}}$$

we need for each $f, g, h \in R$ with $D(f) \equiv D(g) \equiv D(h)$,
a filler of the square by a result due to Kraus [2015]

$$\begin{array}{ccc} R[1/f] & \xrightarrow{\text{sip } \varphi_{fh} \ i} & R[1/h] \\ \| & & \uparrow {\scriptstyle \text{sip } \varphi_{gh} \ j} \\ R[1/f] & \xrightarrow{\text{sip } \varphi_{fg} \ i} & R[1/g] \end{array}$$



$$\Leftrightarrow \quad \text{sip } \varphi_{fh} \equiv \text{sip } \varphi_{fg} \bullet \text{sip } \varphi_{gh}$$

The sheaf property for binary covers

Lemma: $\langle f, g \rangle = A \Rightarrow$ pullback square

$$
\begin{array}{ccc}
A & \longrightarrow & A[1/g] \\
\downarrow & \lrcorner & \downarrow \\
A[1/f] & \longrightarrow & A[1/fg]
\end{array}
$$

The sheaf property for binary covers

$D(h) \equiv D(f) \vee D(g) \;\Rightarrow\; \langle f/1, g/1 \rangle = R[1/h\,]$

$$
\begin{array}{ccc}
R[1/h\,] & \longrightarrow & R[1/h\,][1/g\,] \\
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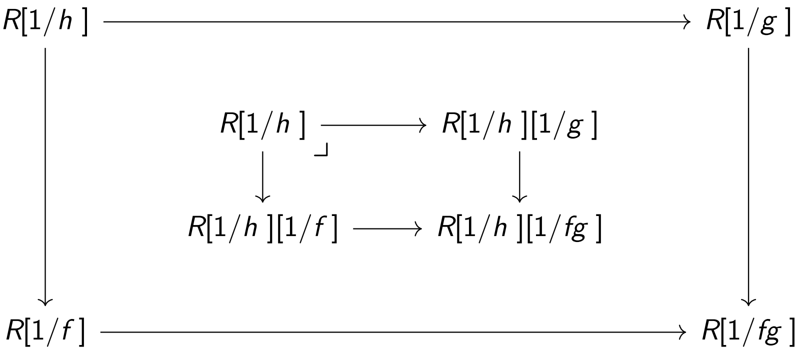

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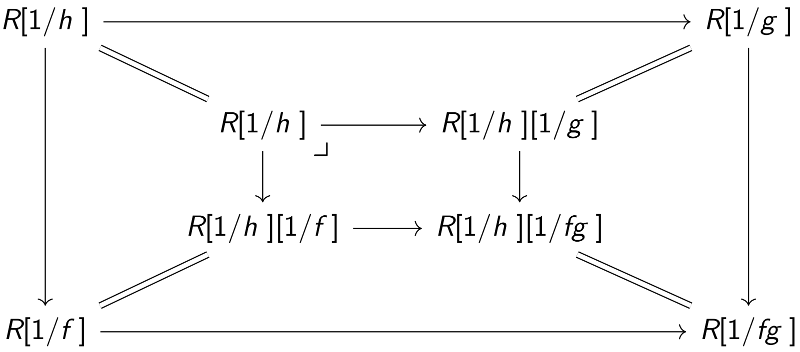

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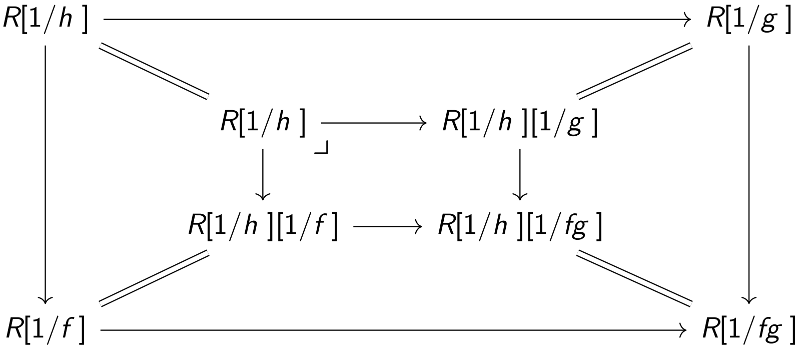

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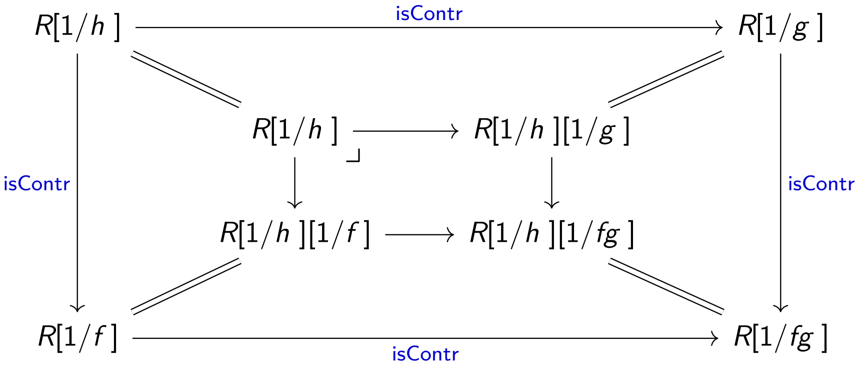

Links to library

- The Zariski lattice
- General construction of presheaf and lemma for sheaf property (line 268)
- Key lemma from univalence
(line 299)

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