Linear Rank Intersection Types

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Intersection Types

Intersection types $\sigma, \sigma_1, \sigma_2, \ldots$ are defined by the following grammar, where $n \geq 1$ and α is a type variable:

$$\sigma ::= \alpha \mid \sigma_1 \cap \cdots \cap \sigma_n \to \sigma.$$

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- In the first intersection type systems, $\cap$ is idempotent.
- **Quantitative types** are the non-idempotent intersection types ($\cap$ is non-idempotent): $\alpha \cap \alpha \rightarrow \beta \neq \alpha \rightarrow \beta$.

Finite Rank

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- Typability is undecidable for unrestricted intersection type systems.
- Restricting intersection types to a finite rank makes typability decidable.

Definition (Rank of intersection types)

Let $\mathbb{T}_0$ be the set of simple types and
$\mathbb{T}_1 = \{\tau_1 \cap \cdots \cap \tau_m \mid \tau_1, \ldots, \tau_m \in \mathbb{T}_0, m \geq 1\}$.

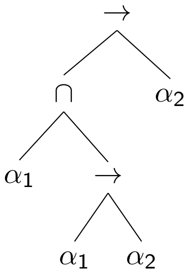
The set $\mathbb{T}_k$, of rank k intersection types (for $k \geq 2$), can be defined recursively in the following way ($n \geq 3$, $m \geq 2$):

$$\mathbb{T}_2 = \mathbb{T}_0 \cup \{\vec{\tau} \to \sigma \mid \vec{\tau} \in \mathbb{T}_1, \sigma \in \mathbb{T}_2\}$$
$$\mathbb{T}_n = \mathbb{T}_{n-1} \cup \{\vec{\tau}_1 \cap \cdots \cap \vec{\tau}_m \to \sigma \mid \vec{\tau}_1, \ldots, \vec{\tau}_m \in \mathbb{T}_{n-1}, \sigma \in \mathbb{T}_n\}$$

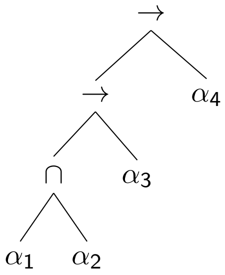
Finite Rank - Example

- The rank of an intersection type is related to the depth of the nested intersections.

$\alpha_1 \cap (\alpha_1 \to \alpha_2) \to \alpha_2$:



$(\alpha_1 \cap \alpha_2 \to \alpha_3) \to \alpha_4$:



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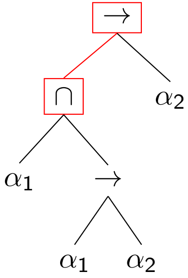
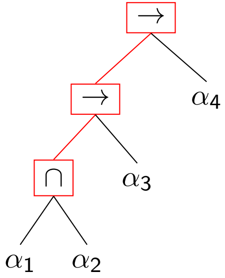


Rank 2

$(\alpha_1 \cap \alpha_2 \to \alpha_3) \to \alpha_4$:



Rank 3

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Consider the term $\lambda fx.f(fx)$.

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- Then, in an idempotent system, it can be typed with: $(\alpha \to \alpha) \to \alpha \to \alpha$.
- This rank decrease makes it possible to type a term like $(\lambda x.x)(\lambda fx.f(fx))$ in a rank 2 idempotent type system, which would not be typable in a rank 2 quantitative type system ($(\lambda x.x)$ must be typed with a rank 3 type).

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Only the linear terms[1] are typed by a simple type in a non-idempotent intersection type system.

[1] Depending on the type system, that can be true for the affine terms.

Linear Rank

- We propose a new definition of rank for intersection types that differs from the previous one in the base case and the introduction of the *linear arrow* $\multimap$.

Definition (Linear rank of intersection types)

Let $\mathbb{T}_{\mathbb{L}0}$ be the set of linear types and
$\mathbb{T}_{\mathbb{L}1} = \{\tau_1 \cap \cdots \cap \tau_m \mid \tau_1, \ldots, \tau_m \in \mathbb{T}_{\mathbb{L}0}, m \geq 1\}$.

The set $\mathbb{T}_{\mathbb{L}k}$, of *linear rank k* intersection types (for $k \geq 2$), can be defined recursively in the following way ($n \geq 3$, $m \geq 2$):

$$
\begin{aligned}
\mathbb{T}_{\mathbb{L}2} = \mathbb{T}_{\mathbb{L}0} &\cup \{\tau \multimap \sigma \mid \tau \in \mathbb{T}_{\mathbb{L}0}, \sigma \in \mathbb{T}_{\mathbb{L}2}\} \\
&\cup \{\tau_1 \cap \cdots \cap \tau_m \to \sigma \mid \tau_1, \ldots, \tau_m \in \mathbb{T}_{\mathbb{L}0}, \sigma \in \mathbb{T}_{\mathbb{L}2}\} \\
\mathbb{T}_{\mathbb{L}n} = \mathbb{T}_{\mathbb{L}n-1} &\cup \{\vec{\tau} \multimap \sigma \mid \vec{\tau} \in \mathbb{T}_{\mathbb{L}n-1}, \sigma \in \mathbb{T}_{\mathbb{L}n}\} \\
&\cup \{\vec{\tau}_1 \cap \cdots \cap \vec{\tau}_m \to \sigma \mid \vec{\tau}_1, \ldots, \vec{\tau}_m \in \mathbb{T}_{\mathbb{L}n-1}, \sigma \in \mathbb{T}_{\mathbb{L}n}\}
\end{aligned}
$$

Linear Rank 2 Intersection Type System

In the Linear Rank 2 Intersection Type System, we say that M has type σ given the environment Γ, and write $\Gamma \vdash_2 M : \sigma$ if it can be obtained from the following *derivation rules*:

$$[x : \tau] \vdash_2 x : \tau \qquad \text{(Axiom)}$$

$$\frac{\Gamma_1, x : \vec{\tau}_1, y : \vec{\tau}_2, \Gamma_2 \vdash_2 M : \sigma}{\Gamma_1, y : \vec{\tau}_2, x : \vec{\tau}_1, \Gamma_2 \vdash_2 M : \sigma} \qquad \text{(Exchange)}$$

$$\frac{\Gamma_1, x_1 : \vec{\tau}_1, x_2 : \vec{\tau}_2, \Gamma_2 \vdash_2 M : \sigma}{\Gamma_1, x : \vec{\tau}_1 \cap \vec{\tau}_2, \Gamma_2 \vdash_2 M[x/x_1, x/x_2] : \sigma} \qquad \text{(Contraction)}$$

$$\frac{\Gamma, x : \tau_1 \cap \cdots \cap \tau_n \vdash_2 M : \sigma \qquad n \geq 2}{\Gamma \vdash_2 \lambda x.M : \tau_1 \cap \cdots \cap \tau_n \to \sigma} \qquad (\to \text{Intro})$$

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Linear Rank 2 Intersection Type System

Structural rules

$$\frac{\Gamma_1, x : \vec{\tau}_1, y : \vec{\tau}_2, \Gamma_2 \vdash_2 M : \sigma}{\Gamma_1, y : \vec{\tau}_2, x : \vec{\tau}_1, \Gamma_2 \vdash_2 M : \sigma} \quad \text{(Exchange)}$$

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Linear Rank 2 Intersection Type System

$\rightarrow$ rules

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- Sound and complete with respect to the Linear Rank 2 Intersection Type System:

Theorem (Soundness)

If $T(M) = (\Gamma, \sigma)$*, then* $\Gamma \vdash_2 M : \sigma$*.*

Theorem (Completeness)

If $\Gamma \vdash_2 M : \sigma$*, then* $T(M) = (\Gamma', \sigma')$ *(for some environment* Γ' *and type* σ'*) and there is a substitution* $\mathbb{S}$ *such that* $\mathbb{S}(\sigma') = \sigma$ *and* $\mathbb{S}(\Gamma') \equiv \Gamma$*.*

Type Inference Algorithm - Example

3. If $M = M_1 M_2$, then:
   - b. if $T(M_1) = (\Gamma'_1, \tau'_1 \cap \cdots \cap \tau'_n \to \sigma'_1)$ (with $n \geq 2$) and, for each $1 \leq i \leq n$, $T(M_2) = (\Gamma_i, \tau_i)$,

     then $T(M) = (\mathbb{S}(\Gamma'_1 + \sum_{i=1}^{n} \Gamma_i), \mathbb{S}(\sigma'_1))$,

     where $\mathbb{S} = UNIFY(\{\tau_i = \tau'_i \mid 1 \leq i \leq n\})$;

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Example

For $M = (\lambda fx.f(fx))(\lambda y.y)$, we have:

- $T(\lambda fx.f(fx)) = ([], ((\alpha_1 \multimap \alpha_2) \cap (\alpha_2 \multimap \alpha_3)) \to \alpha_1 \multimap \alpha_3)$
- $T(\lambda y.y) = ([], \beta_1 \multimap \beta_1) = ([], \beta_2 \multimap \beta_2)$
- $\mathbb{S} = [\alpha_3/\beta_1, \alpha_3/\beta_2, \alpha_3/\alpha_1, \alpha_3/\alpha_2]$

and so $T(M) = (\mathbb{S}([]), \mathbb{S}(\alpha_1 \multimap \alpha_3)) = ([], \alpha_3 \multimap \alpha_3)$.

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- Type inference algorithm that infers the number of reduction steps to normal form.
- In the future, we would like to further explore the relation between our definition of linear rank and the traditional definition of rank, adapt the system and algorithm for other evaluation strategies, and extend them for a programming language.

Thank You!