Sikkel: Multimode Simple Type Theory as an Agda Library

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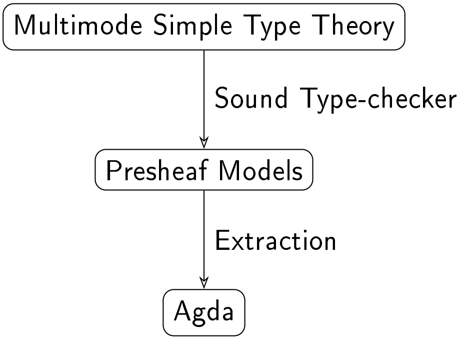
Motivation

Extensions of a standard type theory (MLTT, CIC, ...) with new primitives:

- guarded recursion,
- parametricity,
- univalence,
- directed type theory,
- nominal reasoning,
- ...

How to work in such extended theories within Agda, Coq, ...?

A Brief Overview of Sikkel



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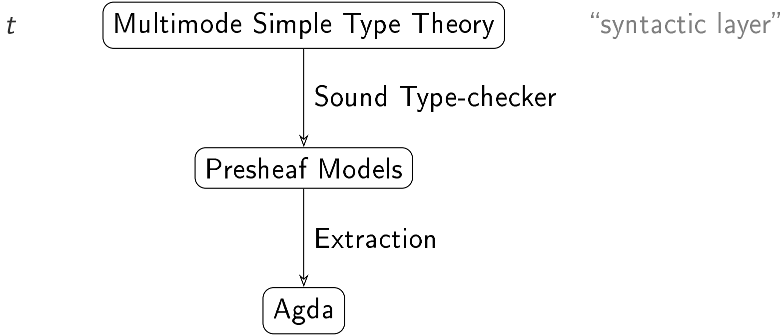

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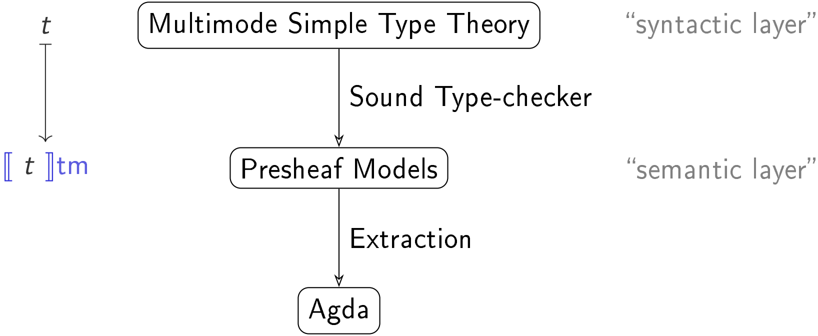

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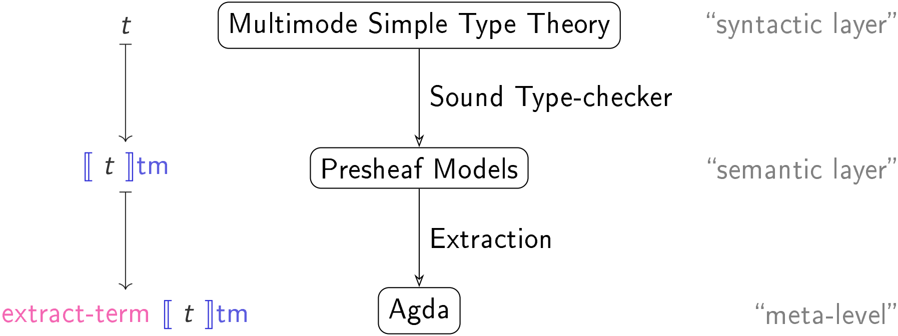

Multimode Simple Type Theory (MSTT)

$\approx$ MTT by Gratzer et al. (LICS20), restricted to simple types.

Parametrized by mode theory ($\approx$ small 2-category):

$$m \xrightarrow{\mu} n$$

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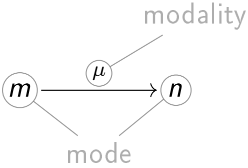

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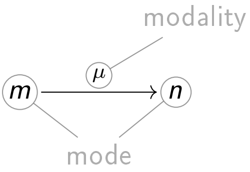

New primitive modal operations:

$$\begin{array}{ccc} \text{CtxExpr } m & \leftarrow & \text{CtxExpr } n \\ \Gamma \text{ ,lock}\langle\, \mu \,\rangle & \leftarrow\!\shortmid & \Gamma \end{array}$$

$$\begin{array}{ccc} \text{TyExpr } m & \rightarrow & \text{TyExpr } n \\ T & \mapsto & \langle\, \mu \mid T \,\rangle \end{array}$$

Programming in Sikkel

$$m \xrightarrow{\mu} n \xrightarrow{\kappa} o$$

Constructing a function of type

$$\langle\ \kappa \mid \langle\ \mu \mid A\ \rangle \Rrightarrow B\ \rangle \Rrightarrow \langle\ \kappa \circledcirc \mu \mid A\ \rangle \Rrightarrow \langle\ \kappa \mid B\ \rangle$$

```
mod-applicative : TmExpr o
mod-applicative =
    { }0
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Hole	Mode	Context	Expected type
0	o	$\diamond$	$\langle\ \kappa \mid \langle\ \mu \mid A\ \rangle \Rrightarrow B\ \rangle \Rrightarrow \langle\ \kappa \circledcirc \mu \mid A\ \rangle \Rrightarrow \langle\ \kappa \mid B\ \rangle$

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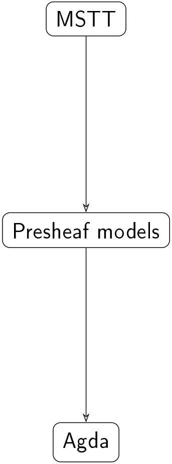
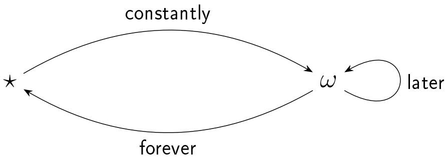
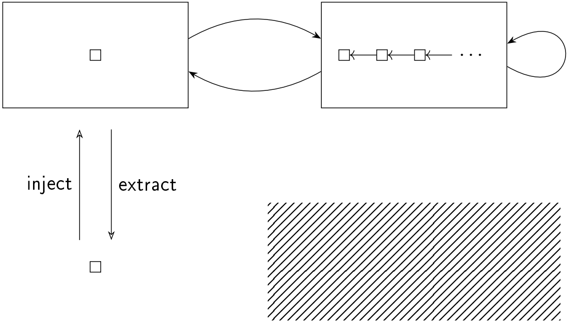
Presheaf Models & Extraction

Application: Guarded Recursion







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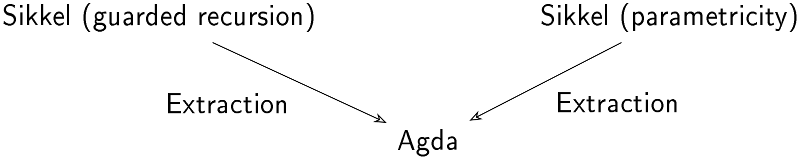
- Completely general in mode theory/base category.
- Elegant extraction mechanism via modalities.
- Modular use of type theory extensions, e.g.



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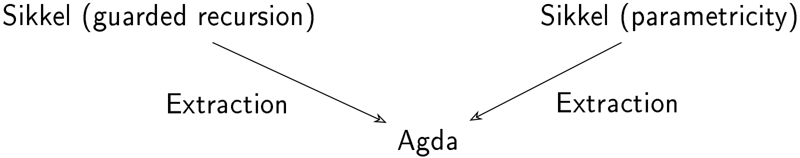
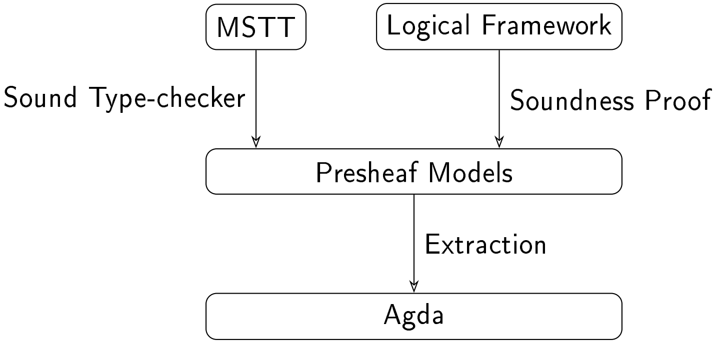


But:

- Custom syntax, so no use of convenient Agda features (e.g. implicits).
- Restricted to MTT-style modal systems, no substructurality.

WIP: Reasoning about Sikkel Programs



Future Work

- Hofmann-Streicher universe at semantic layer.
  - Problems to implement in standard Agda.
  - A variant of Cubical Agda with UIP seems desirable.
- Extension of syntax to full dependent types.
  - Challenging to make interpretation pass termination checker.
- Inductive types (syntactic & semantic layer).
  - E.g. W-types with Sikkel types of constructors and arities.
- Other applications.
  - Ongoing exploration of nominal types.

Thank you for listening!
Questions?

https://github.com/JorisCeulemans/sikkel/releases/tag/v1.0