Unifying Cubical and Multimodal Type Theory

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TYPES, June 2022

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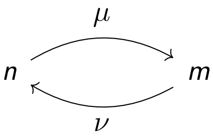

$$\eta : 1_m \to \mu \circ \nu$$

$$\epsilon : \nu \circ \mu \to 1_n$$

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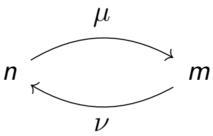

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- Metatheorems: Consistency, normalisation.

Multimodal Type Theory

Rules for modalities (supposing $\mu : n \longrightarrow m$):

$$\frac{\Gamma \ \mathsf{cx} \ @ \ m}{\Gamma, \{\mu\} \ \mathsf{cx} \ @ \ n} \ 🔒$$

$$\frac{\Gamma, \{\mu\} \vdash A \ @ \ n}{\Gamma \vdash \langle \mu \mid A \rangle \ @ \ m} \qquad \frac{\Gamma, \{\mu\} \vdash a : A \ @ \ n}{\Gamma \vdash \mathsf{mod}_{\mu}(a) : \langle \mu \mid A \rangle \ @ \ m}$$

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- Faces and face restrictions (distributive bounded lattice):

$$\frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (r = 0) : \mathbb{F}} \qquad\qquad \frac{\Gamma \vdash \phi : \mathbb{F}}{\Gamma, \phi \ \mathsf{cx}}$$

Cubical Type Theory

- Composition:

$$
\begin{array}{lccc}
\Gamma \vdash & a_0 & & \\
& \| & & \\
\Gamma, \phi \vdash & a & \overset{}{\underset{p}{\rule{4em}{0.4pt}}} & b
\end{array}
$$

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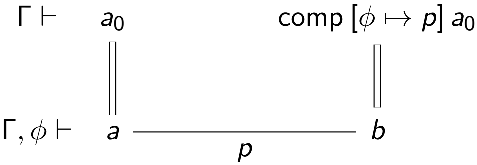

Cubical Type Theory

- Composition:

$$
\begin{array}{lccc}
\Gamma \vdash & a_0 & & \mathsf{comp}\,[\phi \mapsto p]\,a_0 \\
 & \| & & \| \\
\Gamma, \phi \vdash & a & \overset{}{\underset{p}{\rule{5em}{0.4pt}}} & b
\end{array}
$$

- Computation rule for each type.
  - E.g. the composition of a pair is a pair of compositions.

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Principle of Orthogonality

Modal and cubical aspects should interfere minimally with each other.

Exchange Principles

- Exchange operations:

$$\frac{\Gamma \vdash r : \mathbb{I}_m \mathbin{@} m}{\Gamma, \{\mu\} \vdash r^{\mu} : \mathbb{I}_n \mathbin{@} n} \qquad \not\cong$$

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- Exchange substitutions:

$$\frac{}{\Gamma, \{\mu\}, i : \mathbb{I}_n \vdash \sigma^{\mu} : \Gamma, i : \mathbb{I}_m, \{\mu\} \mathrel{@} n} \qquad \cong$$

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- Similar rules for faces.

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- Consequences:
  - Computation rule for composition in modal types.
  - Modal extensionality

$$\langle \mu \mid \mathrm{Path}_A(a, b) \rangle \simeq \mathrm{Path}_{\langle \mu \mid A \rangle}(\mathrm{mod}_{\mu}(a), \mathrm{mod}_{\mu}(b))$$

Presheaf Models

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- Two options:
  - Model each $\mu : n \longrightarrow m$ as the adjunction:

$$\mathbf{PSh}(f(n) \times \square) \underset{(f(\mu) \times \square)^*}{\overset{(f(\mu) \times \square)_*}{\rightleftarrows}} \mathbf{PSh}(f(m) \times \square) \quad (\top)$$

  - Model each $\mu : n \longrightarrow m$ as the adjunction (up to equivalence):

$$\mathbf{PSh}(f(n) \times \square) \underset{(f(\mu) \times \square)_!}{\overset{(f(\mu) \times \square)^*}{\leftrightarrows}} \mathbf{PSh}(f(m) \times \square) \quad (\top)$$

Cubical Multimodal Type Theory

Paper on arXiv: Unifying Cubical and Multimodal Type Theory:

- Full description of Cubical Multimodal Type Theory.
- Modal extensionality.
- Presheaf models.
- Application in guarded recursion.