Proof-relevant normalization for intersection types with profunctors

Types 2022

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Idempotent and non-idempotent intersection types

$$\text{Types } A, B ::= o \mid A \Rightarrow B \mid A \wedge B$$

$a_1 \wedge \cdots \wedge a_n \quad \rightsquigarrow \quad [a_1, \ldots, a_n]$ multiset

λ-term $M \quad \rightsquigarrow \quad [\![M]\!] = \{(\Gamma, a) \mid \Gamma \vdash M : a\}$

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- Idempotent

  $a \wedge a \sim a$

  duplication/erasure

- Non-idempotent

  $a \wedge a \neq a$

  linear resources

Idempotent and non-idempotent intersection types

	Idempotent	Non-idempotent
	$a \wedge a \sim a$	$a \wedge a \neq a$
type	preorder $A = (\lvert A \rvert, \leq_A)$	set $\lvert A \rvert$
	multiset $[a_1, \dots, a_n]$	multiset $[a_1, \dots, a_n]$
	$[a_1, \dots, a_n] \leq_I [b_1, \dots, b_m]$ $\forall i \in \underline{n}, \exists j \in \underline{m}, a_i \leq_A b_j$	$[a_1, \dots, a_n] = [b_1, \dots, b_n]$
term	ideal relation	relation

Idempotent case (Coppo-Dezani)

$$\frac{[a] \leq [a_1, \ldots, a_n]}{\Gamma, x : [a_1, \ldots, a_n] \vdash x : a} \qquad \frac{\Gamma, x : [a_1, \ldots, a_n] \vdash M : a}{\Gamma \vdash \lambda x.M : ([a_1, \ldots, a_n], a)}$$

$$\frac{\Gamma \vdash M : ([a_1, \ldots, a_n], b) \qquad (\Gamma \vdash N : a_i)_{1 \leq i \leq n}}{\Gamma \vdash (M)N : b}$$

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λ-term M with $\mathrm{fv}(M) \subset \vec{x} = \langle x_1, \ldots, x_n \rangle$,

$$[\![M]\!]^{!}_{\vec{x}} \neq \varnothing \qquad \Leftrightarrow \qquad M \text{ head normalizing}$$

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Contraction/weakening $\Rightarrow$ proof using Tait's realizability argument

Non-idempotent case (Gardner-de Carvalho)

$$\frac{}{x : [a] \vdash x : a} \qquad \frac{\Gamma, x : [a_1, \dots, a_n] \vdash M : a}{\Gamma \vdash \lambda x.M : ([a_1, \dots, a_n], a)}$$

$$\frac{\Gamma_0 \vdash M : ([a_1, \dots, a_n], b) \qquad (\Gamma_i \vdash N : a_i)_{1 \leq i \leq n}}{\Gamma_0 + \cdots + \Gamma_n \vdash (M)N : b}$$

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Linear resources $\Rightarrow$ combinatorial proof using derivation size

$\Rightarrow$ quantitative results on computation time

Connecting the two using orthogonality (Ehrhard)

- preorder $A = (|A|, \leq_A)$, subsets $x, y \subseteq |A|$

$$x \perp y \quad :\Leftrightarrow \quad (x \cap y \neq \emptyset \Leftrightarrow \downarrow(x) \cap y \neq \emptyset).$$

- For $D \subseteq \mathcal{P}(|A|)$, define $D^{\perp} := \{y \in \mathcal{P}(|A|) \mid \forall x \in D, x \perp y\}$

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$\rightsquigarrow$ new category where both typing systems coexist

objects	(A, D) preorder A $+$ extra structure $D = D^{\perp\perp}$
morphisms	ideal relations preserving the extra structure

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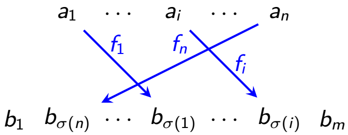
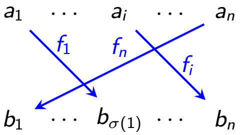
- We obtain:

$$[\![M]\!]^{\mathbf{NI}}_{\vec{x}} \neq \varnothing \qquad \Leftrightarrow \qquad [\![M]\!]^{\mathbf{I}}_{\vec{x}} \neq \varnothing$$

From preorders to categories

preorder $A = (\|A\|, \leq_A)$	category $\mathbf{A}$
$a \leq_A b$	$a \xrightarrow{f} b$
multiset $[a_1, \ldots, a_n]$	list $\langle a_1, \ldots, a_n \rangle \in \mathcal{C}\mathbf{A}$
$[a_1, \ldots, a_n] \leq [b_1, \ldots, b_m]$	$\langle a_1, \ldots, a_n \rangle \to \langle b_1, \ldots, b_m \rangle$
$\forall i \in \underline{n}, \exists j \in \underline{m}, a_i \leq_A b_j$	
$[a, a] \leq [a]$ and $[a, a] \leq [a]$	$\langle a, a \rangle \rightleftarrows \langle a \rangle$

From sets to groupoids

set $(|A|, =)$	groupoid $\mathbf{A}$
$a = b$	$a \xrightarrow{\cong} b$
multiset $[a_1, \dots, a_n]$	list $\langle a_1, \dots, a_n \rangle \in \mathcal{L}\mathbf{A}$
	$\langle a_1, \dots, a_n \rangle \to \langle b_1, \dots, b_n \rangle$
$[a_1, \dots, a_n] = [b_1, \dots, b_n]$	
$[a, a] \neq [a]$	$\mathcal{L}\mathbf{A}(\langle a, a \rangle, \langle a \rangle) = \varnothing$

From relations to profunctors

$$R \subseteq A \times B \quad \Leftrightarrow \quad A \xrightarrow{\textit{function}} \mathcal{P}(B) \quad \Leftrightarrow \quad A \times B \xrightarrow{\textit{function}} 2$$

Definition

Let **A** and **B** be two categories, a *profunctor* from **A** to **B** is a functor

$$P : \mathbf{A} \to \widehat{\mathbf{B}} \qquad (\text{also denoted } P : \mathbf{A} \nrightarrow \mathbf{B})$$

$$P : \mathbf{A} \nrightarrow \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A} \xrightarrow{\textit{functor}} \widehat{\mathbf{B}} \quad \Leftrightarrow \quad \mathbf{A} \times \mathbf{B}^{\mathrm{op}} \xrightarrow{\textit{functor}} \mathbf{Set}$$

Profunctor composition being not strictly associative, we need to work in the setting of bicategories

Intersection type profunctors (Olimpieri)

Reflexive object $\mathbf{D} \cong \mathbf{D} \Rightarrow \mathbf{D}$ in the bicategory **Prof**

General idea: for a λ-term M with $\mathrm{fv}(M) \subset \vec{x} = \langle x_1, \ldots, x_n \rangle$,

$$[\![M]\!]_{\vec{x}} \quad \text{is a profunctor} \quad (!\mathbf{D})^n \nrightarrow \mathbf{D}$$

$$[\![M]\!]_{\vec{x}}(\Gamma, a) \quad = \quad \left\{ \begin{array}{c} \pi \\ \vdots \\ \hline \Gamma \vdash M : a \end{array} \right\}_{/\sim}$$

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Question: what can the bicategorical setting tell us about the 2-cells?

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Linear resources: explicit reduction $2-$cell

$$[\![M]\!]^{\mathbf{NI}}_{\vec{x}} \quad \Rightarrow \quad [\![HN(M)]\!]^{\mathbf{NI}}_{\vec{x}}$$

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Contraction/weakening: proof using Tait's realizability argument

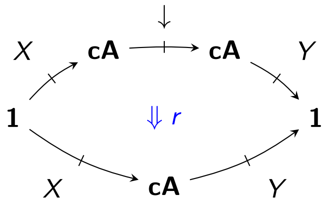
Bicategorical orthogonality

- A preorder with underlying set $|A|$

$$(x, y) \in \perp_A \quad :\Leftrightarrow \quad (x \cap y \neq \emptyset \Leftrightarrow {\downarrow}(x) \cap y \neq \emptyset).$$

- $\mathbf{A}$ category with groupoid core $\mathbf{cA}$

$$(X, Y, r) \in \perp_{\mathbf{A}} \quad :\Leftrightarrow \quad r \text{ retract to the canonical inclusion}$$



Orthogonality bicategory

$\rightsquigarrow$ new bicategory where both typing systems coexist

types	$(\mathbf{A}, \mathbf{D})$ category $+$ extra structure
terms	profuntors preserving the extra structure
reductions	natural transformations preserving the extra structure

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We obtain: for a term M, there are injection retraction 2-cells:

$$\downarrow [\![M]\!]^{\mathbf{NI}} \quad \leftleftarrows\!\!\!\rightrightarrows \quad [\![M]\!]^{\mathbf{I}}$$

Future work

- Study execution time in the idempotent setting
- General theory of orthogonality bicategories

Thank you