Transpension: The Right Adjoint to the Π-type

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Presheaf semantics can model:

- **HoTT** (preservation of **isomorphisms**),
- **Parametricity** (preservation of **relations**),
- **Guarded TT** (preservation of **stage of computation**),
- **Nominal TT** (preservation of **renaming** and α**-equivalence**),
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Use these preservation properties **within type theory**?

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Internalization Operators

We want **simpler foundations:**

- simpler metatheory,
- cross-fertilization (e.g. non-affine Ψ/Gel),
- guidance (e.g. directed TT).

Transpension		Strictness	Pushout
$\forall u \dashv \between u$		(OP18)	
(Yet87, ND22)			

HoTT $(\mathbb{I} \to \sqcup) \dashv \surd$ (LOPS18)			HoTT/DirTT/Param. Glue (CCHM16/ NVD17/WL20)	Param. Weld (NVD17)
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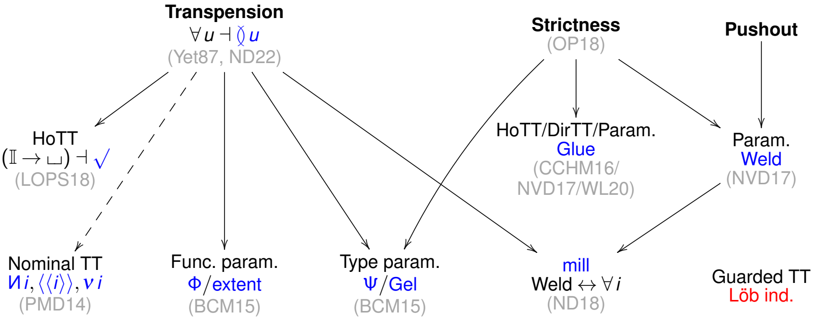

LMCS submission: `https://arxiv.org/abs/2008.08533`

- Type system with transpension $\between u$ for shape variable $u : \mathbb{U}$, e.g.
  - $i : \mathbb{I}$ in HoTT/Param.,
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MTT is parametrized by a **2-category**:

- modes $p, q, r, \ldots$
- modalities $\mu : p \to q$,
  - On types $T \mapsto \langle \mu \mid T \rangle$,
  - On contexts $\Gamma \mapsto (\Gamma, 🔒_\mu)$,
  - $🔒_\mu \dashv \mu$.
- (2-cells $\alpha : \mu \Rightarrow \nu$).

Typical **semantics:**

- $[\![p]\!]$ is a presheaf category modelling all of DTT,
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Pick a base category $\mathscr{W}$.
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- **Problem:** Modalities $\forall(u : \mathbb{U}) \dashv \between u$ bind / depend on u (unsupported by MTT).
- **Solution: Modes** are shape contexts
  e.g. $\Xi = (u : \mathbb{U}, v : \mathbb{V}, w : \mathbb{W})$
  Formally, Ξ is a context in $\mathrm{Psh}(\mathscr{W})$.
- **Judgements at mode** Ξ are modelled in $\mathrm{Psh}(\int_{\mathscr{W}} \Xi)$:

TT in $\mathrm{Psh}(\int_{\mathscr{W}} \Xi)$		TT in $\mathrm{Psh}(\mathscr{W})$
		Ξ ctx
Γ ctx	$\sim$	$\Xi.\Gamma$ ctx
$\Gamma \vdash T$ type	$\sim$	$\Xi.\Gamma \vdash T$ type
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Shapes/Multipliers

- A **(substructural?) shape** is an $U \in \mathscr{W}$ together with an **arbitrary** "multiplier" functor $\sqcup \ltimes U : \mathscr{W} \to \mathscr{W}$ such that $U \cong \top \ltimes U$.

  E.g. twisted prism functor (PK19)

- A **cartesian shape** is a $U \in \mathscr{W}$ where $\mathscr{W}$ is cartesian.
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Multiplier extends over $\mathbf{y} : \mathscr{W} \subseteq \mathrm{Psh}(\mathscr{W})$ as $\sqcup \ltimes \mathbb{U} : \mathrm{Psh}(\mathscr{W}) \to \mathrm{Psh}(\mathscr{W}) : \Xi \mapsto (\Xi, u : \mathbb{U})$ (**shape context extension**).

Assume multiplier is **local right adjoint**:

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$$\exists^{\int \Xi}_{\mathbb{U}} \dashv \reflectbox{$\exists$}^{\int \Xi}_{\mathbb{U}} : \int_{\mathscr{W}} \Xi \to \int_{\mathscr{W}} (\Xi, u : \mathbb{U}),$$

whence 4 presheaf functors and 3 modalities

$$\begin{array}{ccccccc}
(\exists^{\int \Xi}_{\mathbb{U}})_! & \dashv & (\exists^{\int \Xi}_{\mathbb{U}})^* & \dashv & (\exists^{\int \Xi}_{\mathbb{U}})_* & & \\
 & & \| \wr & & \| \wr & & \\
 & & (\reflectbox{$\exists$}^{\int \Xi}_{\mathbb{U}})_! & \dashv & (\reflectbox{$\exists$}^{\int \Xi}_{\mathbb{U}})^* & \dashv & (\reflectbox{$\exists$}^{\int \Xi}_{\mathbb{U}})_* \\
\hline\hline
\exists^{\Xi}_{\mathbb{U}} & \dashv & \reflectbox{$\exists$}^{\Xi}_{\mathbb{U}} & \dashv & \forall^{\Xi}_{\mathbb{U}} & \dashv & \between^{\Xi}_{\mathbb{U}} \\
\hline\hline
[\![\text{🔒}_{\reflectbox{$\exists$} u}]\!] & \dashv & [\![\text{🔒}_{\forall u}]\!] & \dashv & [\![\text{🔒}_{\between u}]\!] & & \\
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Shape Weakening Modalities

From $\pi_1 : (\Xi, u : \mathbb{U}) \to \Xi$ whence

$$\Sigma_{\mathbb{U}}^{\int \Xi} : \int_{\mathscr{W}} (\Xi, u : \mathbb{U}) \to \int_{\mathscr{W}} \Xi,$$

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$$(\Sigma_{\mathbb{U}}^{\int \Xi})_! \dashv (\Sigma_{\mathbb{U}}^{\int \Xi})^* \dashv (\Sigma_{\mathbb{U}}^{\int \Xi})_*$$

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Shape weakening $\Omega\, u$ is a modality (does not compute).
For **cartesian** multipliers, we get $Ⅎ\, u = \Omega\, u$ and $\forall\, u = \Pi\, u \dashv \between\, u$.

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Shape Substitution Modalities

From $\sigma : \Xi_1 \to \Xi_2$ (e.g. weakening), we get

$$\Sigma^{\int \sigma} : \int_{\mathscr{W}} \Xi_1 \to \int_{\mathscr{W}} \Xi_2,$$

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Conclusion

We can recover most internal presheaf operators using:

- The **transpension type**,
- The **strictness axiom**,
- A **pushout type**.

Basic semantics of transpension are straightforward.

Take home message

Check the preprint! `https://arxiv.org/abs/2008.08533`

Thanks!

Questions?

Conclusion

We can recover most internal presheaf operators using:

- The **transpension type**,
- The **strictness axiom**,
- A **pushout type**.

Basic semantics of transpension are straightforward.

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Questions?

Quantification Theorem

- For **semicartesian** multipliers (where $\Omega\, u$ exists), we get

$$\mathsf{spoil}_u : \text{⅃}\, u \Rightarrow \Omega\, u \qquad \mathsf{cospoil}_u : \sqcap\, u \Rightarrow \forall\, u$$

- For **cartesian** multipliers, we get

$$\text{⅃}\, u = \Omega\, u \qquad \sqcap\, u = \forall\, u$$

- For **cancellative and affine** multipliers (where ⅃_U is **fully faithful**), we get invertible (co-)units

$$\mathsf{const}_u : 1 \cong \forall\, u \circ \text{⅃}\, u$$

$$\mathsf{unmer}_u : \forall\, u \circ \between\, u \cong 1$$

- For **cancellative, affine and connection-free** multipliers, we get Φ/extent.

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Definition

$\mathbb{U}$ is **atomic** if $\mathbb{U} \to -$ has a right adjoint $\sqrt[\mathbb{U}]{-}$.

Theorem

…iff $\Pi(u : \mathbb{U})$ has a right adjoint $\between u$.
(∇ in Yet87)

Theorem

Representables (e.g. $\mathbb{I}$) are atomic.

If $\mathbb{U}$ is atomic, then

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Counterexample

Bool is not atomic.

Higher-Dimensional Pattern Matching

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Counterexample

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Examples

Constructor for ${}^{\Omega u \dashv}\Pi\, u$:

$$\frac{\Xi, u : \mathbb{U} \mid \Omega\, u(\Gamma) \vdash a : A}{\Xi \mid \Gamma \vdash \mathrm{mod}_{\Pi u}\, a : \langle \Pi\, u \mid A \rangle}$$

Projection by co-unit:
$\mathrm{app}_{v/u} : (\Omega\, v \shortmid \langle \Pi\, u \mid A \rangle) \to A[\mathrm{copy}_{v/u}\uparrow]$

Constructor for ${}^{\Finv u \dashv}\forall\, u$:

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Constructor for $^{ⅎ u \dashv}\forall\, u$:

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Projection by co-unit:
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