Towards Probabilistic Reasoning about Typed Combinatory Terms

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1. Logic of Combinatory Logic (LCL)

2. Probabilistic Reasoning about Typed Combinatory Terms (PCL)

Outline

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2. Probabilistic Reasoning about Typed Combinatory Terms (PCL)

LCL Syntax

- Logic of combinatory logic (*LCL*) - classical propositional logic over simply typed combinatory logic

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Type assignment statement

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Syntax of *LCL*

$$\alpha, \beta := M : \sigma \mid \neg\alpha \mid \alpha \Rightarrow \beta$$

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(Ax 1) $\mathsf{S} : (\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$
(Ax 2) $\mathsf{K} : \sigma \to (\tau \to \sigma)$
(Ax 3) $\mathsf{I} : \sigma \to \sigma$
(Ax 4) $(M : \sigma \to \tau) \Rightarrow ((N : \sigma) \Rightarrow (MN : \tau))$
(Ax 5) $M : \sigma \Rightarrow N : \sigma$, if $M = N$
(Ax 6) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$
(Ax 7) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$
(Ax 8) $(\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\alpha \Rightarrow \beta) \Rightarrow \neg\alpha)$

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(Ax 7) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$
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- Inference rule:

$$\frac{\alpha \Rightarrow \beta \qquad \alpha}{\beta}\ (\mathsf{MP})$$

LCL Semantics

Applicative structure for *LCL*

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Soundness and completeness of the axiomatiation for *LCL*

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Completeness of Ax

If $T \models \alpha$, then $T \vdash \alpha$.

Simona Kašterović and Silvia Ghilezan. Logic of combinatory logic, February 2021. Submitted for publication

Outline

PCL

- *PCL* - a probabilistic system for simply typed combinatory terms;
- Probabilistic logic over *LCL*;
- Pribabilistic logic LPP_2

$$\phi := P_{\geq s}\alpha \mid \neg\phi \mid \phi \wedge \phi,$$

where α is a formula of classical propositional logic and $s \in [0,1] \cap \mathbb{Q}$.
- *PCL* syntax

$$\phi := P_{\geq s}\alpha \mid \neg\phi \mid \phi \wedge \phi,$$

where α is an *LCL*-formula and $s \in [0,1] \cap \mathbb{Q}$.
- $P_{\geq s}\alpha$ has a meaning "probability that α is true is greater than or equal to s"

Zoran Ognjanović, Miodrag Rašković, and Zoran Marković. Probability Logics - Probability-Based Formalization of Uncertain Reasoning. Springer, 2016.

PCL Axiomatization

LCL axiomatization + axiomatic system for probability logic

Axiomatic system for probability logic:

(1) all instances of the classical propositional tautologies, (atoms are any *PCL*-formulas),

(2) $P_{\geq 0}\alpha$,

(3) $P_{\leq r}\alpha \Rightarrow P_{<s}\alpha$, $s > r$,

(4) $P_{<s}\alpha \Rightarrow P_{\leq s}\alpha$,

(5) $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \Rightarrow P_{\geq min\{1,r+s\}}(\alpha \vee \beta)$,

(6) $(P_{\leq r}\alpha \wedge P_{<s}\beta) \Rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$,

(7) $P_{\geq 1}(\alpha \Rightarrow \beta) \Rightarrow (P_{\geq s}\alpha \Rightarrow P_{\geq s}\beta)$.

PCL Semantics - an idea

$$\mathcal{M} = \langle W, \{\rho_w\}_w, H, \mu \rangle$$

- W is a nonempty set of worlds, where each world is one LCL-applicative structure;
- $\rho_w : V \times \{w\} \to D_w$;
- H is an algebra of subsets of W;
- μ is a finitely additive probability measure defined on H.

S. Ghilezan, J. Ivetić, S. Kašterovć, Z. Ognjanović, and N. Savić. Towards probabilistic reasoning in type theory - the intersection type case. 11th International Symposium, FoIKS 2020, Dortmund, Germany, February 17-21, 2020, Proceedings, volume 12012 of Lecture Notes in Computer Science, pages 122–139. Springer, 2020.

S. Ghilezan, J. Ivetić, S. Kašterović, Z. Ognjanović, and N. Savić. Probabilistic reasoning about typed lambda terms. International Symposium, LFCS 2018, Deerfield Beach, FL, USA, January 8-11, 2018, Proceedings, volume 10703 of Lecture Notes in Computer Science, pages 170–189. Springer, 2018

Goals

- Prove soundness;
- Prove completeness.

Thank you for your attention!

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