Conservativity of Two-Level Type Theory Corresponds to Staged Compilation

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Overview

Two-level TT:

- *Voevodsky: A simple type system with two identity types*
- *Annekov, Capriotti, Kraus, Sattler: Two-Level Type Theory and Applications*
- Goal: synthetic homotopy theory

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- Template Haskell, MetaOCaml
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- *Remark: staged compilation $\neq$ staged computation*

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5. *Splicing:* for $t : \Uparrow A$, we have $\sim t : A$.

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5. *Splicing:* for $t : \Uparrow A$, we have $\sim t : A$.
6. Quoting and splicing are definitional inverses.

Inlined definitions

Staging input:

$$\mathsf{two} : {\Uparrow}\mathsf{Nat}_0$$
$$\mathsf{two} = <\mathsf{suc}_0\,(\mathsf{suc}_0\,\mathsf{zero}_0)>$$

$$\mathsf{f} : \mathsf{Nat}_0 \to \mathsf{Nat}_0$$
$$\mathsf{f} = \lambda\, x.\, x + {\sim}\mathsf{two}$$

Inlined definitions

Staging input:

$$\begin{aligned}&\mathsf{two} : \Uparrow\mathsf{Nat}_0\\&\mathsf{two} = <\mathsf{suc}_0\,(\mathsf{suc}_0\,\mathsf{zero}_0)>\end{aligned}$$

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Output:

$$\begin{aligned}&\mathsf{f} : \mathsf{Nat}_0 \to \mathsf{Nat}_0\\&\mathsf{f} = \lambda\, x.\, x + \mathsf{suc}_0\,(\mathsf{suc}_0\,\mathsf{zero}_0)\end{aligned}$$

Compile-time functions

Input:

$$\mathsf{id} : (A : \mathsf{U}_1) \to A \to A$$
$$\mathsf{id} = \lambda\, A\, x.\, x$$

$$\mathsf{idBool}_0 : \mathsf{Bool}_0 \to \mathsf{Bool}_0$$
$$\mathsf{idBool}_0 = \lambda\, x.\, {\sim}(\mathsf{id}\, {<}\mathsf{Bool}_0{>}\, {<}x{>})$$

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$$\mathsf{idBool}_0 = \lambda\, x.\, {\sim}(\mathsf{id} < \mathsf{Bool}_0 > < x >)$$

Output:

$$\mathsf{idBool}_0 : \mathsf{Bool}_0 \to \mathsf{Bool}_0$$
$$\mathsf{idBool}_0 = \lambda\, x.\, x$$

Inlined map arguments

Input:

$$\mathsf{inlMap} : \{A\,B : \Uparrow\mathsf{U}_0\} \to (\Uparrow{\sim}A \to \Uparrow{\sim}B) \to \Uparrow(\mathsf{List}_0\,{\sim}A) \to \Uparrow(\mathsf{List}_0\,{\sim}B)$$
$$\mathsf{inlMap} = \lambda\,f\;as.\,{<}\mathsf{foldr}_0\,(\lambda\,a\,bs.\,\mathsf{cons}_0\,{\sim}(f\;{<}a{>})\;bs)\;\mathsf{nil}_0\;{\sim}as{>}$$

$$\mathsf{f} : \mathsf{List}_0\,\mathsf{Nat}_0 \to \mathsf{List}_0\,\mathsf{Nat}_0$$
$$\mathsf{f} = \lambda\,xs.\;{\sim}(\mathsf{inlMap}\,(\lambda\,n.\,{<}{\sim}n + 2{>})\,{<}xs{>})$$

Inlined map arguments

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$$\mathsf{inlMap} : \{A\,B : \Uparrow\mathsf{U}_0\} \to (\Uparrow\!\sim\! A \to \Uparrow\!\sim\! B) \to \Uparrow(\mathsf{List}_0 \sim\! A) \to \Uparrow(\mathsf{List}_0 \sim\! B)$$
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$$\mathsf{f} = \lambda\, xs.\, \mathsf{foldr}_0\,(\lambda\, a\; bs.\, \mathsf{cons}_0\,(a + 2)\; bs)\; \mathsf{nil}_0\; xs$$

Staging Types

Input:

$$
\begin{aligned}
&\mathsf{Vec} : \mathsf{Nat}_1 \to {\Uparrow}\mathsf{U}_0 \to {\Uparrow}\mathsf{U}_0 \\
&\mathsf{Vec}\,\mathsf{zero}_1 \quad\;\; A = <\top_0> \\
&\mathsf{Vec}\,(\mathsf{suc}_1\, n)\, A = <\sim A \times_0 \sim(\mathsf{Vec}\, n\, A)>
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{Tuple3} : \mathsf{U}_0 \to \mathsf{U}_0 \\
&\mathsf{Tuple3}\, A = \sim(\mathsf{Vec}\, 3 <A>)
\end{aligned}
$$

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$$\mathsf{Vec} : \mathsf{Nat}_1 \to {\Uparrow}\mathsf{U}_0 \to {\Uparrow}\mathsf{U}_0$$
$$\mathsf{Vec}\,\mathsf{zero}_1 \quad A = <\top_0>$$
$$\mathsf{Vec}\,(\mathsf{suc}_1\ n)\,A = <{\sim}A \times_0 {\sim}(\mathsf{Vec}\,n\,A)>$$

$$\mathsf{Tuple3} : \mathsf{U}_0 \to \mathsf{U}_0$$
$$\mathsf{Tuple3}\,A = {\sim}(\mathsf{Vec}\,3\,<A>)$$

Output:

$$\mathsf{Tuple3} : \mathsf{U}_0 \to \mathsf{U}_0$$
$$\mathsf{Tuple3}\,A = A \times_0 (A \times_0 (A \times_0 \top_0))$$

map for Vec

Input:

$$\begin{aligned}
&\mathsf{map} : \{A\,B : \Uparrow\mathsf{U}_0\} \to (n : \mathsf{Nat}_1) \to (\Uparrow{\sim}A \to \Uparrow{\sim}B)\\
&\qquad\qquad\qquad\qquad \to \Uparrow(\mathsf{Vec}\,n\,A) \to \Uparrow(\mathsf{Vec}\,n\,B)\\
&\mathsf{map}\,\mathsf{zero}_1 \qquad f\ as = {<}\mathsf{tt}_0{>}\\
&\mathsf{map}\,(\mathsf{suc}_1\,n)\,f\ as = {<}({\sim}(f\,{<}\mathsf{fst}_0\,{\sim}as{>}),\ {\sim}(\mathsf{map}\,n\,f\,{<}\mathsf{snd}_0\,{\sim}as{>})){>}
\end{aligned}$$

$$\begin{aligned}
&\mathsf{f} : {\sim}(\mathsf{Vec}\,2\,{<}\mathsf{Nat}_0{>}) \to {\sim}(\mathsf{Vec}\,2\,{<}\mathsf{Nat}_0{>})\\
&\mathsf{f}\,xs = {\sim}(\mathsf{map}\,2\,(\lambda\,x.\,{<}{\sim}x + 2{>})\,{<}xs{>})
\end{aligned}$$

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Input:

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\begin{aligned}
&\mathsf{map} : \{A\,B : \Uparrow\mathsf{U}_0\} \to (n : \mathsf{Nat}_1) \to (\Uparrow{\sim}A \to \Uparrow{\sim}B)\\
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&\mathsf{map}\,\mathsf{zero}_1 \quad\;\; f\;as = <\mathsf{tt}_0>\\
&\mathsf{map}\,(\mathsf{suc}_1\;n)\;f\;as = <({\sim}(f\;<\mathsf{fst}_0\;{\sim}as>),\;{\sim}(\mathsf{map}\,n\,f\;<\mathsf{snd}_0\;{\sim}as>))>
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&\mathsf{f}\;xs = {\sim}(\mathsf{map}\,2\,(\lambda\,x.\,<{\sim}x + 2>)\,<xs>)
\end{aligned}
$$

Output:

$$
\begin{aligned}
&\mathsf{f} : \mathsf{Nat}_0 \times_0 (\mathsf{Nat}_0 \times_0 \top_0) \to \mathsf{Nat}_0 \times_0 (\mathsf{Nat}_0 \times_0 \top_0)\\
&\mathsf{f}\;xs = (\mathsf{fst}_0\,xs + 2,\;(\mathsf{fst}_0\,(\mathsf{snd}_0\,xs) + 2,\;\mathsf{tt}_0))
\end{aligned}
$$

Ergonomics

In the demo implementation:

- Bidirectional elaboration
- Coercive subtyping for $\Uparrow$ and type formers
- Standard unification techniques

Almost all quotes and splices are inferable in practice.

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The **object-level fragment** of 2LTT contains types in U_0, their terms, and only allows contexts with entries in U_0.

Conservativity of 2LTT means

- There's a bijection between object-theoretic types and object-fragment 2LTT types.
- There's also a bijection between object-theoretic terms and object-fragment 2LTT terms.
- (Both up to $\beta\eta$-conversion).

(See proof in the preprint)

ICFP preprint, implementation, tutorial:
github.com/AndrasKovacs/staged

Thanks for your attention!