More on modal embeddings and calling paradigms

José Espírito Santo\textsuperscript{1} \hspace{1cm} Luís Pinto\textsuperscript{1} \hspace{1cm} Tarmo Uustalu\textsuperscript{2}

\textsuperscript{1}University of Minho
\textsuperscript{2}Reykjavik University and Tallinn University of Technology

TYPES 2022
Nantes, 23 June 2022
Overview of the talk

1. Introduction and recap of calling paradigms
2. Modal target and embeddings
3. More on embeddings and paradigms
Modal embeddings

From Troelstra-Schwichtenberg, “Basic Proof Theory”

9.2 Embedding intuitionistic logic into S4

9.2.1. DEFINITION. (The modal embedding). The embedding exists in several variants. We describe a variant $\circ$, and a more familiar variant $\Box$. The definition is by induction on the depth of formulas ($P$ atomic, not $\bot$):

\begin{align*}
P^\circ & := P \\
\bot^\circ & := \bot \\
(A \land B)^\circ & := A^\circ \land B^\circ \\
(A \lor B)^\circ & := \Box A^\circ \lor \Box B^\circ \\
(A \rightarrow B)^\circ & := \Box A^\circ \rightarrow B^\circ \\
(\exists x A)^\circ & := \exists x A^\circ \\
(\forall x A)^\circ & := \forall x A^\circ
\end{align*}

9.2.2. PROPOSITION. The two versions of the modal embedding are equivalent in the following sense: $S4 \vdash \Box A^\circ \iff A^\circ$ and hence $S4 \vdash \Box \Gamma^\circ \rightarrow A^\circ$. 

9.2.3. A (remarkable) fact is that, if $\Gamma^\circ$ is a consistent set of formulas, then $\Gamma^\circ \cup \{A^\circ\}$ is also consistent. 

9.2.4. A (remarkable) fact is that, if $\Gamma^\circ$ is a consistent set of formulas, then $\Gamma^\circ \cup \{A^\circ\}$ is also consistent. 

9.2.5. A (remarkable) fact is that, if $\Gamma^\circ$ is a consistent set of formulas, then $\Gamma^\circ \cup \{A^\circ\}$ is also consistent.
Calling paradigms

Plotkin, 1975: over the same syntax

- a “programming language” (with notions of value and evaluation)
- a $\lambda$-calculus (with notions of reduction and equality)
- the two components linked by a standardization theorem
Calling paradigms

Plotkin, 1975: over the same syntax

- a “programming language” (with notions of value and evaluation)
- a \(\lambda\)-calculus (with notions of reduction and equality)
- the two components linked by a standardization theorem

This is what we call a calling paradigm
Calling paradigms

Plotkin, 1975: over the same syntax

- a “programming language” (with notions of value and evaluation)
- a $\lambda$-calculus (with notions of reduction and equality)
- the two components linked by a standardization theorem

This is what we call a calling paradigm

Two examples

- Call-by-value (Plotkin’s $\lambda$-calculus)
- Call-by-name (ordinary $\lambda$-calculus)
Calling paradigms

Plotkin, 1975: over the same syntax

- a “programming language” (with notions of value and evaluation)
- a $\lambda$-calculus (with notions of reduction and equality)
- the two components linked by a standardization theorem

This is what we call a calling paradigm

Two examples

- Call-by-value (Plotkin’s $\lambda$-calculus)
- Call-by-name (ordinary $\lambda$-calculus)

We will talk about a third one

- Call-by-box
Computational interpretation of the modal embeddings

We consider implication only

- Gödel’s embedding: \( A \supset B \iff \Box A \supset \Box B \)
- Girard’s embedding: \( A \supset B \iff \Box A \supset B \)
We consider implication only

- Gödel’s embedding: $A \supset B \leftrightarrow \Box A \supset \Box B$
- Girard’s embedding: $A \supset B \leftrightarrow \Box A \supset B$

Analogous interpretations into linear logic suggest

- Gödel’s embedding interprets call-by-value
- Girard’s embedding interprets call-by-name
Our approach

- Focus on the interpretation into $S4$
- Identify the modal target appropriate for the interpretation
- Identify the core of the connection embeddings-paradigms
- Later instantiations of the modality are a separate concern
- Decompose other known interpretations as the composition of a modal embedding and some instantiation of the modality
Modal embeddings and later instantiations

\[ \lambda_n \quad \text{Girard} \quad \lambda_v \quad \text{Gödel} \]

modal target
Modal embeddings and later instantiations

\[ \lambda_n \xrightarrow{\text{Girard}} \text{modal target} \xrightarrow{\text{instantiation}} \text{some system} \]

\[ \lambda_v \xrightarrow{\text{Gödel}} \]
Modal embeddings and later instantiations

Several possible instantiations:
Modal embeddings and later instantiations

Several possible instantiations: \( \Box A \rightleftharpoons !A \)
Modal embeddings and later instantiations

Several possible instantiations: $\square A = T \supset A$
Modal embeddings and later instantiations

Several possible instantiations: $\Box A = FUA$
Call-by-name: $\lambda_n$

(Terms) \[ M, N ::= x \mid \lambda x. M \mid MN \]

($\beta_n$) \[ (\lambda x. M)N \rightarrow [N/x]M \]
Call-by-name: $\lambda_n$

\[(Terms) \quad M, N ::= x \mid \lambda x. M \mid MN\]

\[(\beta_n) \quad (\lambda x. M)N \rightarrow [N/x]M\]

\[
\begin{align*}
M \rightarrow M' & \quad (\mu) \\
MN \rightarrow M'N
\end{align*}
\]

\[
\begin{align*}
N \rightarrow N' & \quad (\nu) \\
MN \rightarrow MN'
\end{align*}
\]

\[
\begin{align*}
M \rightarrow M' & \quad (\xi) \\
\lambda x. M \rightarrow \lambda x. M'
\end{align*}
\]

Call-by-name reduction and equality: $\rightarrow_{\beta_n}^*$ and $=_{\beta_n}$

$\rightarrow_n$: $\beta_n$ closed under $\mu$

Call-by-name evaluation: $\rightarrow_n^*$
Introduction

Modal-target-embeddings

More on this

Standardization for cbn

Standard reducibility $M \Rightarrow_n N$:

\[
\begin{align*}
& x \Rightarrow_n x & \text{VAR} \\
& \lambda x. M \Rightarrow_n \lambda x. N & \text{ABS} \\
& M \Rightarrow_n M' & N \Rightarrow_n N' & \text{APL}
\end{align*}
\]

\[
\begin{align*}
& M \rightarrow_n^* \lambda x. M' & [N/x] M' \Rightarrow_n P & \text{RDX}
\end{align*}
\]

Standardization theorem: $M \rightarrow_n^* N$ iff $M \Rightarrow_n N$
Call-by-value: $\lambda_v$

\[
\begin{align*}
\text{(Terms)} \quad M, N &::= V | MN \\
\text{(Values)} \quad V &::= x | \lambda x. M
\end{align*}
\]

\[
\begin{align*}
(\beta_v) \quad (\lambda x. M) V &\rightarrow [V/x] M \\
(\mu) \quad MN &\rightarrow M'N \\
(\nu_{\text{val}}) \quad VN &\rightarrow VN' \\
(\xi) \quad \lambda x. M &\rightarrow \lambda x. M'
\end{align*}
\]

Call-by-value reduction and equality: $\rightarrow^*_{\beta_v}$ and $=_{\beta_v}$

$\rightarrow_v$: $\beta_v$ closed under $\mu$ and $\nu_{\text{val}}$

Call-by-value evaluation: $\rightarrow^*_v$
Standard reducibility $M \Rightarrow_v N$:

\[
\begin{align*}
\text{VAR} & \quad M \Rightarrow_v N \\
\lambda x. M & \Rightarrow_v \lambda x. N \\
\text{ABS} & \quad M \Rightarrow_v M' \quad N \Rightarrow_v N' \\
MN & \Rightarrow_v M'N' \\
\text{APL} & \quad M \rightarrow_v^* \lambda x. M' \quad N \rightarrow_v^* V \\
[ V / x ] M' & \Rightarrow_v P \\
MN & \Rightarrow_v P \\
\text{RDX} & \quad M \rightarrow_v^* N \text{ iff } M \Rightarrow_v N
\end{align*}
\]

Standardization theorem: $M \rightarrow_{v \beta}^* N$ iff $M \Rightarrow_v N$
MODAL TARGET

AND

MODAL EMBEDDINGS
Modal target: the box calculus $\lambda_b$

\[ M, N ::= \epsilon(x) \mid \lambda x. M \mid MN \mid \text{box}(M) \]

\[(\beta_b) \quad (\lambda x. M)\text{box}(N) \rightarrow [N/\epsilon(x)]M\]
Modal target: the box calculus $\lambda_b$

$$M, N ::= \varepsilon(x) \mid \lambda x. M \mid MN \mid \text{box}(M)$$

$$(\beta_b) \ (\lambda x. M)\text{box}(N) \rightarrow [N/\varepsilon(x)]M$$

- Call-by-box reduction: $\rightarrow^{*}_{\beta_b}$
- $\rightarrow_{we}$: forbid reduction under $\lambda$-abstraction and boxes
- Weak and external reduction, or cbb evaluation: $\rightarrow^{*}_{we}$
Modal target: the box calculus $\lambda_b$

$$A ::= X \mid B \supset A \mid B \quad B ::= \Box A$$

Contexts $\Gamma$ are sets of declarations $x : B$

$$\Gamma, x : \Box A \vdash \varepsilon(x) : A \quad \Gamma \vdash \text{box}(M) : \Box A$$

$$\Gamma, x : B \vdash M : A \quad \Gamma \vdash M : B \supset A \quad \Gamma \vdash N : B$$

$$\Gamma \vdash \lambda x. M : B \supset A \quad \Gamma \vdash MN : A$$
Calling paradigm cbb

Standard reducibility $M \Rightarrow_b N$:

\[
\begin{align*}
\epsilon(x) & \Rightarrow_b \epsilon(x) \quad \text{VAR} \\
\lambda x. M & \Rightarrow_b \lambda x. N \quad \text{ABS} \\
M & \Rightarrow_b M' \quad N & \Rightarrow_b N' \\
MN & \Rightarrow_b M'N' \quad \text{APL} \\
\text{box}(M) & \Rightarrow_b \text{box}(N) \\
M & \rightarrow^{*}_{we} \lambda x. M' \quad N & \rightarrow^{*}_{we} \text{box}(N') \\
MN & \Rightarrow_b [N'/\epsilon(x)]M' \Rightarrow_b P \quad \text{RDX}
\end{align*}
\]
Calling paradigm cbb

Standard reducibility $M \Rightarrow_b N$:

\[
\begin{align*}
\frac{\varepsilon(x) \Rightarrow_b \varepsilon(x)}{\text{VAR}} & \quad \frac{M \Rightarrow_b N}{\lambda x.M \Rightarrow_b \lambda x.N} & \quad \text{ABS} \\
M \Rightarrow_b M' \quad N \Rightarrow_b N' & \quad \frac{MN \Rightarrow_b M'N'}{\text{APL}} & \quad \frac{M \Rightarrow_b N}{\text{box}(M) \Rightarrow_b \text{box}(N)} \quad \text{BOX} \\
M \rightarrow^*_\text{we} \lambda x.M' \quad N \rightarrow^*_\text{we} \text{box}(N') & \quad \frac{\left[N'/\varepsilon(x)\right]M' \Rightarrow_b P}{MN \Rightarrow_b P} \quad \text{RDX}
\end{align*}
\]

Theorem (Standardization for $\lambda_b$)

$M \rightarrow^*_\beta_b N$ iff $M \Rightarrow_b N$
Girard’s embedding: \( \lambda_n \rightarrow \lambda_b \)

Translation of formulas

\[
X^\circ = X \\
(A_1 \supset A_2)^\circ = \Box A_1^\circ \supset A_2^\circ
\]

Translation of terms

\[
x^\circ = \varepsilon(x) \\
(\lambda x. M)^\circ = \lambda x. M^\circ \\
(MN)^\circ = M^\circ \text{box}(N^\circ)
\]

Typing

\[
\Gamma \vdash M : A \text{ in } \lambda_n \text{ iff } \Box \Gamma^\circ \vdash M^\circ : A^\circ \text{ in } \lambda_b
\]
Girard’s embedding: $\lambda_n \rightarrow \lambda_b$

**Theorem (Properties of Girard’s translation)**

1. *(Preservation and reflection of reduction)* $M \rightarrow_{\beta_n} N$ in $\lambda_n$ iff $M^\circ \rightarrow_{\beta_b} N^\circ$ in $\lambda_b$.

2. *(Preservation and reflection of evaluation)* $M \rightarrow_n N$ in $\lambda_n$ iff $M^\circ \rightarrow_{we} N^\circ$ in $\lambda_b$.

3. *(Preservation and reflection of standard reduction)* $M \Rightarrow_n N$ in $\lambda_n$ iff $M^\circ \Rightarrow_b N^\circ$ in $\lambda_b$.

**Corollary**

*Standardization for $\lambda_n$.***
Gödel’s embedding: $\lambda_v \rightarrow \lambda_b$

Translation of formulas

$$A^* = \Box A^*$$

$$X^* = X$$

$$(A_1 \supset A_2)^* = \Box A_1^* \supset \Box A_2^*$$

Translation of terms

$$V^* = \text{box}(V^*)$$

$$(MN)^* = \text{raise}(N^*)M^*$$

$$x^* = \varepsilon(x)$$

$$(\lambda x. M)^* = \lambda x. M^*$$

where

$$\text{raise}(N) := \lambda z. \varepsilon(z)N$$
Gödel’s embedding: $\lambda_v \rightarrow \lambda_b$

**Typing**

- $\Gamma \vdash M : A$ in $\lambda_v$ iff $\Gamma^* \vdash M^* : A^*$ in $\lambda_b$
- $\Gamma \vdash V : A$ in $\lambda_v$ iff $\Gamma^* \vdash V^* : A^*$ in $\lambda_b$

Notice the derived typing rule

$$
\begin{align*}
\Gamma & \vdash N : B \\
\Gamma & \vdash \text{raise}(N) : (\Box(B \supset B')) \supset B'
\end{align*}
$$
Gödel’s embedding: $\lambda_v \rightarrow \lambda_b$

Theorem (Properties of Gödel’s translation)

1. (Preservation and reflection of reduction) $M \rightarrow_{\beta_v} N$ in $\lambda_v$ iff $M^* \rightarrow_{\beta_{b2}} N^*$ in $\lambda_b$.

2. (Preservation and reflection of evaluation) $M \rightarrow^*_v V$ in $\lambda_v$ iff $M^* \rightarrow^*_w V^*$ in $\lambda_b$.

3. (Preservation and reflection of standard red.) $M \Rightarrow_v N$ in $\lambda_v$ iff $M^* \Rightarrow_{b2} N^*$ in $\lambda_b$. 
Gödel’s embedding: \( \lambda_v \rightarrow \lambda_b \)

**Theorem (Properties of Gödel’s translation)**

1. *(Preservation and reflection of reduction)* \( M \rightarrow_{\beta_v} N \) in \( \lambda_v \) iff \( M^* \rightarrow_{\beta_{b2}} N^* \) in \( \lambda_b \).

2. *(Preservation and reflection of evaluation)* \( M \rightarrow^*_{v} V \) in \( \lambda_v \) iff \( M^* \rightarrow^*_{we} V^* \) in \( \lambda_b \).

3. *(Preservation and reflection of standard red.)* \( M \Rightarrow_{\nu} N \) in \( \lambda_v \) iff \( M^* \Rightarrow_{b2} N^* \) in \( \lambda_b \).

**Corollary**

*Standardization for \( \lambda_v \).*
The extra bits in the cbv case

In $\lambda_b$ define $\beta_{b_2} \subset\rightarrow^{2}_{\text{we}}$:

$$\text{raise}(\text{box}(N))\text{box}(\lambda x. P) \rightarrow [N/\varepsilon(x)]P$$

In $\lambda_b$ define $\Rightarrow_{b_2} \subset\Rightarrow_{b}$, by replacing

$$M \rightarrow^{*}_{\text{we}} \lambda x. M' \quad N \rightarrow^{*}_{\text{we}} \text{box}(N') \quad [N'/\varepsilon(x)]M' \Rightarrow_{b} P$$

RDX

with

$$N \rightarrow^{*}_{\text{we}} \text{box}(N') \quad M \rightarrow^{*}_{\text{we}} \text{box}(\lambda x. M') \quad [N'/\varepsilon(x)]M' \Rightarrow_{b_2} Q$$

RDX2
The modal target is a new calling paradigm, call-by-box (cbb)
The modal embeddings implement “protect-by-a-box”, an abstract form of the compilation technique “protect-by-a-lambda”
“Protect-by-a-box” has a cbn side and a cbv side
The modal embeddings unify cbn and cbv inside cbb

\(^{1}\)JES, L. Pinto, T. Uustalu, “Modal embeddings and calling paradigms”, FSCD 2019
MORE ONE EMBEDDINGS AND PARADIGMS
Needed

- Improve the properties of Gödel’s embedding
- Change the narrative, to fit better with the strong properties of Girard’s embedding
- Separation of the extra bits by modal reasons
Redesign the modal target

Starting from $\lambda_b$

- Forbid types of the form $\Box\Box A$
- Separate, already in the untyped syntax, terms that must have a modal type from those that cannot have a modal type
- Application will remain an ambiguous constructor: separate the two forms (and the corresponding reduction rules)
- Just keep the particular forms of application that show up in the image of the embeddings
Types, terms and reduction

(Types) \[ A ::= B | C \]
(Boxed types) \[ B ::= □ C \]
(Unboxed types) \[ C ::= X | B ⊃ A \]
Types, terms and reduction

(Types) \[ A ::= B \mid C \]

(Boxed types) \[ B ::= \Box C \]

(Unboxed types) \[ C ::= X \mid B \supset A \]

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \epsilon(x) \mid \lambda x. T \mid M@_b Q \]

(Boxed terms) \[ P, Q ::= \text{box}(M) \mid M@_b Q \]

\[ V \]

[\{ z \}]}
Types, terms and reduction

(Types) \( A ::= B \mid C \)

(Boxed types) \( B ::= □C \)

(Unboxed types) \( C ::= X \mid B \supset A \)

(Terms) \( T ::= M \mid P \)

(Unboxed terms) \( M, N ::= \varepsilon(x) \mid \lambda x. T \mid M@_b Q \)

(Boxed terms) \( P, Q ::= \text{box}(M) \mid M@_b Q \)

\[(\lambda x. T)@\text{box}(N) \rightarrow [N/\varepsilon(x)]M\]
Derived notion of application and reduction

\[
\begin{align*}
\Gamma, z : \Box (B' \supset B) & \vdash \varepsilon(z) : B' \supset B \\
\Gamma, z : \Box (B' \supset B) & \vdash Q : B' \\
\Gamma, z : \Box (B' \supset B) & \vdash \varepsilon(z)@b Q : B \\
\Gamma & \vdash \lambda z . \varepsilon(z)@b Q : (\Box (B' \supset B)) \supset B \\
\Gamma & \vdash P : \Box (B' \supset B) \\
\Gamma & \vdash (\lambda z . \varepsilon(z)@b Q)@b P : B \\
\end{align*}
\]
Derived notion of application and reduction

\[
\begin{align*}
\Gamma, z : \Box (B' \supset B) & \vdash \varepsilon(z) : B' \supset B & \Gamma, z : \Box (B' \supset B) & \vdash Q : B' \\
\Gamma, z : \Box (B' \supset B) & \vdash \varepsilon(z) \@_{b} Q : B & W & \\
\Gamma & \vdash \lambda z . \varepsilon(z) \@_{b} Q : (\Box (B' \supset B)) \supset B & \\
\Gamma & \vdash (\lambda z . \varepsilon(z) \@_{b} Q) \@_{b} P : B & \Gamma & \vdash \Box (B' \supset B) & \quad \Gamma & \vdash P : \Box (B' \supset B) & \quad \Gamma & \vdash QP
\end{align*}
\]

box\((N)(box(\lambda x. P)) \rightarrow [N/\varepsilon(x)]P\)
The final syntax

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \{ x \} \mid \varepsilon(x) \mid \lambda x. T \mid M @ b \mid Q \]

(Boxed terms) \[ P, Q ::= P \mid box(M) \mid Q \]

\[ V \]

\[ B \]
The final syntax

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= [V] \epsilon(x) \mid \lambda x. T \mid M \otimes_b Q \]

(Boxed terms) \[ P, Q ::= [B] \text{box}(M) \mid QP \]
The final syntax

(Terms) \( T ::= M \mid P \)

(Unboxed terms) \( M, N ::= \{ \varepsilon(x) \mid \lambda x. T \mid MQ \}

(Boxed terms) \( P, Q ::= \{ \text{box}(M) \mid QP \} \)
The final syntax

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \{ \varepsilon(x) \mid \lambda x. T \mid MQ \} \]

(Boxed terms) \[ P, Q ::= \{ \text{box}(M) \mid QP \} \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \mid Q_m \cdots Q_1B \]
The final syntax: another box calculus $\lambda_{\Box}$

(Terms) $T ::= M | P$

(Unboxed terms) $M, N ::= \epsilon(x) | \lambda x. T | MQ$

(Boxed terms) $P, Q ::= \text{box}(M) | QP$

The general form of (un)boxed terms

$$VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 \mathcal{B}$$

Two reduction rules

$$(\beta_<) \quad (\lambda x. M)\text{box}(N) \rightarrow [N/\epsilon(x)]M$$

$$(\beta_> \quad \text{box}(N)(\text{box}(\lambda x. P)) \rightarrow [N/\epsilon(x)]P$$
Girard’s translation: choose the unboxed mode
Girard’s translation: choose the unboxed mode

(Terms) \[ T ::= M | P \]

(Unboxed terms) \[ M, N ::= \epsilon(x) | \lambda x. T | MQ \]

(Boxed terms) \[ P, Q ::= box(M) | QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \]

\[ Q_m \cdots Q_1 B \]

Two reduction rules

(\(\beta_\prec\)) \[(\lambda x.M)box(N) \rightarrow [N/\epsilon(x)]M\]

(\(\beta_\succ\)) \[ box(N)(box(\lambda x.P)) \rightarrow [N/\epsilon(x)]P \]
Girard’s translation: choose the unboxed mode

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \varepsilon(x) \mid \lambda x. T \mid MQ \]

(Boxed terms) \[ P, Q ::= \text{box}(M) \mid QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 B \]

Two reduction rules

\[
\begin{align*}
(\beta_<) & \quad (\lambda x. M)\text{box}(N) \rightarrow [N/\varepsilon(x)]M \\
(\beta_> ) & \quad \text{box}(N)(\text{box}(\lambda x. P)) \rightarrow [N/\varepsilon(x)]P
\end{align*}
\]
Girard’s translation: choose the unboxed mode

(Terms) \[ T ::= M | P \]

(Unboxed terms) \[ M, N ::= ε(x) | \lambda x. M | MQ \]

(Boxed terms) \[ P, Q ::= \text{box}(M) | QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1B \]

Two reduction rules

\[ \begin{align*}
(β_<) \quad (\lambda x. M)\text{box}(N) & \rightarrow [N/ε(x)]M \\
(β_> ) \quad \text{box}(N)(\text{box}(\lambda x. P)) & \rightarrow [N/ε(x)]P
\end{align*} \]
Girard’s translation: choose the unboxed mode

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \varepsilon(x) \mid \lambda x. M \mid M_{box}(N) \]

(Boxed terms) \[ P, Q ::= box(M) \mid QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 B \]

Two reduction rules

\[ (\beta_<) \quad (\lambda x. M)_{box}(N) \rightarrow [N/\varepsilon(x)]M \]
\[ (\beta_>\ ) \quad box(N)(box(\lambda x. P)) \rightarrow [N/\varepsilon(x)]P \]
Girard’s translation: choose the unboxed mode

\[(\text{T}erms) \quad T \ ::= \ M \mid P\]

\[(\text{Unboxed terms}) \quad M, N \ ::= \ \varepsilon(x) \mid \lambda x. M \mid M_{\text{box}}(N)\]

\[(\text{Boxed terms}) \quad P, Q \ ::= \ \text{box}(M) \mid QP\]

The general form of (un)boxed terms

\[VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 B\]

Two reduction rules

\[\beta_< \quad (\lambda x. M)_{\text{box}}(N) \rightarrow [N/\varepsilon(x)] M\]

\[\beta_> \quad \text{box}(N)(\text{box}(\lambda x. P)) \rightarrow [N/\varepsilon(x)] P\]
Girard’s translation: choose the unboxed mode

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \varepsilon(x) \mid \lambda x. M \mid M_{\text{box}}(N) \]

(Boxed terms) \[ P, Q ::= \text{box}(M) \mid QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \]

\[ Q_m \cdots Q_1 B \]

Two reduction rules

\[ (\beta_<) \quad (\lambda x. M)_{\text{box}}(N) \rightarrow [N/\varepsilon(x)]M \]

\[ (\beta_> \quad \text{box}(N)(\text{box}(\lambda x. P)) \rightarrow [N/\varepsilon(x)]P \]
Girard’s translation: choose the unboxed mode

(Terms) \( T ::= M | P \)

(Unboxed terms) \( M, N ::= \epsilon(x) | \lambda x. M | M \text{box}(N) \)

(Boxed terms) \( P, Q ::= \text{box}(M) | QP \)

The general form of (un)boxed terms

\( VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 B \)

Two reduction rules

\[
\begin{align*}
(\beta_<) & \quad (\lambda x. M) \text{box}(N) \rightarrow [N/\epsilon(x)] M \\
(\beta_> ) & \quad \text{box}(N)(\text{box}(\lambda x. P)) \rightarrow [N/\epsilon(x)] P
\end{align*}
\]
Gödel’s translation: choose the boxed mode

Terms

\[ T ::= M | P \]

Unboxed terms

\[ M, N ::= \{ z \} | \{ \} \epsilon (x) | \lambda x. T | MQ \]

Boxed terms

\[ P, Q ::= \text{box}(M) | \{ z \} B | QP \]

The general form of (un)boxed terms

\[ VQ_1 \ldots Q_m Q_m \ldots Q_1 B \]

Two reduction rules

(\beta < 0)

\[(\lambda x. M)\text{box}(N) \rightarrow \left[ N/\epsilon(x) \right] M\]

(\beta > 0)

\[
\text{box}(N)(\text{box}(\lambda x. P)) \rightarrow \left[ N/\epsilon(x) \right] P
\]
Gödel’s translation: choose the boxed mode

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \varepsilon(x) \mid \lambda x. T \mid MQ \]

(Boxed terms) \[ P, Q ::= \text{box}(M) \mid QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1B \]

Two reduction rules

\[ (\beta_<) \quad (\lambda x.M)\text{box}(N) \rightarrow [N/\varepsilon(x)]M \]

\[ (\beta_>) \quad \text{box}(N)(\text{box}(\lambda x.P)) \rightarrow [N/\varepsilon(x)]P \]
Gödel’s translation: choose the boxed mode

(Terms) \[ T ::= M \mid P \]

(Unboxed terms) \[ M, N ::= \varepsilon(x) \mid \lambda x. T \mid MQ \]

(Boxed terms) \[ P, Q ::= \text{box}(M) \mid QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 B \]

Two reduction rules

\[ (\beta_<) \quad (\lambda x. M)\text{box}(N) \rightarrow [N/\varepsilon(x)]M \]
\[ (\beta_>) \quad \text{box}(N)(\text{box}(\lambda x. P)) \rightarrow [N/\varepsilon(x)]P \]
Gödel’s translation: choose the boxed mode

\[
(Terms) \quad T ::= M \mid P
\]

\[
(Unboxed \ terms) \quad M, N ::= \varepsilon(x) \mid \lambda x.P \mid MQ
\]

\[
(Boxed \ terms) \quad P, Q ::= \text{box}(M) \mid QP
\]

The general form of (un)boxed terms

\[\begin{array}{l}
VQ_1 \cdots Q_m \\
Q_m \cdots Q_1 B
\end{array}\]

Two reduction rules

\[
\begin{array}{ll}
(\beta_<) & (\lambda x.M)\text{box}(N) \rightarrow [N/\varepsilon(x)]M \\
(\beta_> \ ) & \text{box}(N)(\text{box}(\lambda x.P)) \rightarrow [N/\varepsilon(x)]P
\end{array}
\]
Gödel’s translation: choose the boxed mode

(Terms) \( T ::= M | P \)

(Unboxed terms) \( V ::= \varepsilon(x) | \lambda x. P | MQ \)

(Boxed terms) \( P, Q ::= \text{box}(V) | QP \)

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 B \]

Two reduction rules

\[ (\beta_<) \quad (\lambda x. M)\text{box}(N) \rightarrow [N/\varepsilon(x)] M \]
\[ (\beta_> \quad \text{box}(N)(\text{box}(\lambda x. P)) \rightarrow [N/\varepsilon(x)] P \]
Gödel’s translation: choose the boxed mode

(Terms) \[ T ::= M | P \]

(Unboxed terms) \[ V ::= ε(x) | λx.P | MQ \]

(Boxed terms) \[ P, Q ::= \text{box}(V) | QP \]

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1B \]

Two reduction rules

\[ (β_<) \quad (λx.M)\text{box}(N) \rightarrow [N/ε(x)]M \]
\[ (β_> ) \quad \text{box}(N)(\text{box}(λx.P)) \rightarrow [N/ε(x)]P \]
Gödel’s translation: choose the boxed mode

(Terms) \( T ::= M | P \)

(Unboxed terms) \( V ::= \varepsilon(x) | \lambda x.P | MQ \)

(Boxed terms) \( P, Q ::= \text{box}(V) | QP \)

The general form of (un)boxed terms

\[ VQ_1 \cdots Q_m \quad Q_m \cdots Q_1 B \]

Two reduction rules

\[
\begin{align*}
(\beta_\prec) \quad & \quad (\lambda x.M)\text{box}(N) & \rightarrow & \quad [N/\varepsilon(x)]M \\
(\beta_\succ) \quad & \quad \text{box}(N)(\text{box}(\lambda x.P)) & \rightarrow & \quad [N/\varepsilon(x)]P
\end{align*}
\]
Results

Theorem (Properties of Girard’s translation from $\lambda_n$ to $\lambda_\otimes$)

1. $M \rightarrow_{\beta_n} N$ in $\lambda_n$ iff $M^\circ \rightarrow_{\beta} N^\circ$ in $\lambda_\otimes$.
2. $M \rightarrow^*_{n} N$ in $\lambda_n$ iff $M^\circ \rightarrow_{b} N^\circ$ in $\lambda_\otimes$.
3. $M \Rightarrow_{n} N$ in $\lambda_n$ iff $M^\circ \Rightarrow_{\otimes} N^\circ$ in $\lambda_\otimes$.

Theorem (Properties of Gödel’s translation from $\lambda_v$ to $\lambda_\otimes$)

1. $M \rightarrow_{\beta_v} N$ in $\lambda_v$ iff $M^* \rightarrow_{\beta} N^*$ in $\lambda_\otimes$.
2. $M \rightarrow^*_{v} N$ in $\lambda_v$ iff $M^* \rightarrow_{b} N^*$ in $\lambda_\otimes$.
3. $M \Rightarrow_{v} N$ in $\lambda_v$ iff $M^* \Rightarrow_{\otimes} N^*$ in $\lambda_b$. 
The modal target still follows call-by-box and combines two “modes” of the application constructor and reduction.

Each mode corresponds to a calling paradigm cbn or cbv.

The modal embeddings point out isomorphic copies of cbn or cbn inside cbb.

Cbn and Cbv coexist inside cbb.

\(^2\)JES, L. Pinto, T. Uustalu, “Plotkin’s lambda-calculus as a modal calculus”, JLAMP, 2022
Work in progress

- Explore the redesigned modal calculus $\lambda_\Rightarrow$
- Work out fully the several instantiations
THANK YOU