More on modal embeddings and calling paradigms

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Overview of the talk

1. Introduction and recap of calling paradigms
2. Modal target and embeddings
3. More on embeddings and paradigms

Modal embeddings

From Troelstra-Schwichtenberg, “Basic Proof Theory”

9.1.5B. ♠ Formulate a ... only K-axioms, no T- and 4-axioms. ... this system.

9.2 Embedding intuitionistic logic into S4

9.2.1. Definition. (*The modal embedding*). The embedding exists in several variants. We describe a variant $^\circ$, and a more familiar variant $^\Box$. The definition is by induction on the depth of formulas (P atomic, not $\bot$):

$$
\begin{array}{llllll}
P^\circ & := & P & P^\Box & := & \Box P \\
\bot^\circ & := & \bot & \bot^\Box & := & \bot \\
(A \wedge B)^\circ & := & A^\circ \wedge B^\circ & (A \wedge B)^\Box & := & A^\Box \wedge B^\Box \\
(A \vee B)^\circ & := & \Box A^\circ \vee \Box B^\circ & (A \vee B)^\Box & := & A^\Box \vee B^\Box \\
(A \to B)^\circ & := & \Box A^\circ \to B^\circ & (A \to B)^\Box & := & \Box(A^\Box \to B^\Box) \\
(\exists x A)^\circ & := & \exists x \Box A^\circ & (\exists x A)^\Box & := & \exists x A^\Box \\
(\forall x A)^\circ & := & \forall x A^\circ & (\forall x A)^\Box & := & \Box \forall x A^\Box
\end{array}
$$

⊠

9.2.2. Proposition. *The two versions of the modal embedding are equivalent in the following sense:* $\mathbf{S4} \vdash \Box A^\circ \leftrightarrow A^\Box$ *and hence* $\mathbf{S4} \vdash \Box\Gamma^\circ \Rightarrow A^\circ$ *iff* $\mathbf{S4} \vdash \Gamma^\Box \Rightarrow A^\Box$. ⊠

9.2.2A. ♠ ...

Calling paradigms

Plotkin, 1975: over the same syntax

- a "programming language" (with notions of value and evaluation)
- a λ-calculus (with notions of reduction and equality)
- the two components linked by a standardization theorem

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Two examples

- Call-by-value (Plotkin's λ-calculus)
- Call-by-name (ordinary λ-calculus)

Calling paradigms

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Two examples

- Call-by-value (Plotkin’s λ-calculus)
- Call-by-name (ordinary λ-calculus)

We will talk about a third one

- Call-by-box

Computational interpretation of the modal embeddings

We consider implication only

- Gödel's embedding: $A \supset B \mapsto \Box A \supset \Box B$
- Girard's embedding: $A \supset B \mapsto \Box A \supset B$

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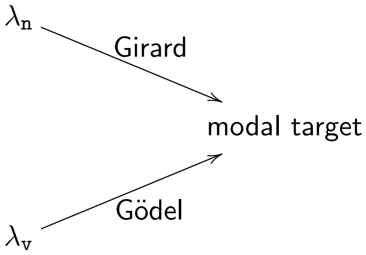
Analogous interpretations into linear logic suggest

- Gödel's embedding interprets call-by-value
- Girard's embedding interprets call-by-name

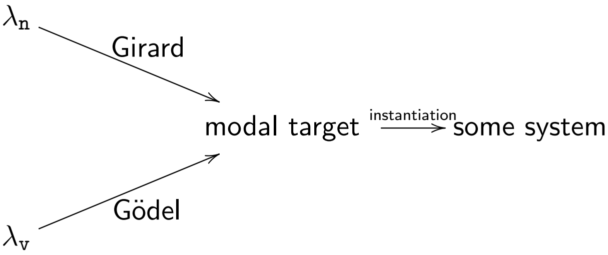
Our approach

- Focus on the interpretation into $S4$
- Identify the modal target appropriate for the interpretation
- Identify the core of the connection embeddings-paradigms
- Later instantiations of the modality are a separate concern
- Decompose other known interpretations as the composition of a modal embedding and some instantiation of the modality

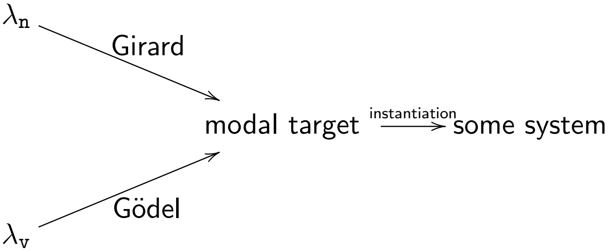
Modal embeddings and later instantiations



Modal embeddings and later instantiations

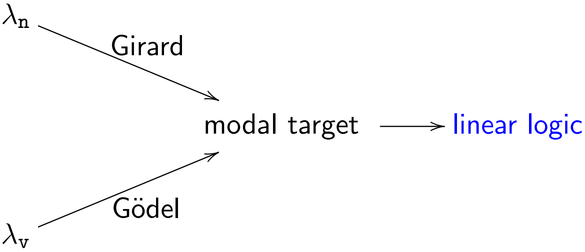


Modal embeddings and later instantiations



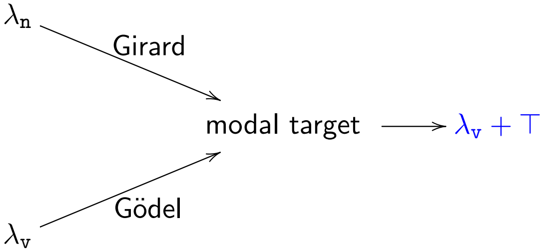
Several possible instantiations:

Modal embeddings and later instantiations



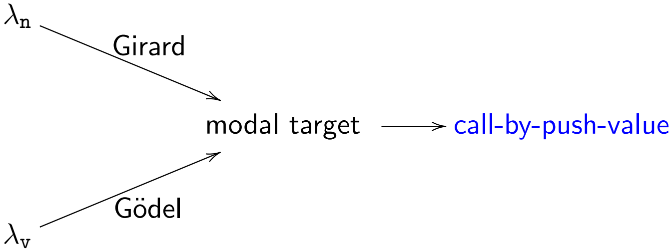
Several possible instantiations: $\Box A = !A$

Modal embeddings and later instantiations



Several possible instantiations: $\Box A = \top \supset A$

Modal embeddings and later instantiations



Several possible instantiations: $\Box A = FUA$

Call-by-name: $\lambda_{\mathtt{n}}$

$$(\mathit{Terms}) \quad M, N ::= x \mid \lambda x.M \mid MN$$

$$(\beta_{\mathtt{n}}) \qquad (\lambda x.M)N \to [N/x]M$$

Call-by-name: $\lambda_{\mathtt{n}}$

$$(\mathit{Terms}) \quad M, N ::= x \mid \lambda x.M \mid MN$$

$$(\beta_{\mathtt{n}}) \qquad (\lambda x.M)N \to [N/x]M$$

$$\frac{M \to M'}{MN \to M'N}\ (\mu) \qquad \frac{N \to N'}{MN \to MN'}\ (\nu) \qquad \frac{M \to M'}{\lambda x.M \to \lambda x.M'}\ (\xi)$$

Call-by-name reduction and equality: $\to^*_{\beta_{\mathtt{n}}}$ and $=_{\beta_{\mathtt{n}}}$

$\to_{\mathtt{n}}$: $\beta_{\mathtt{n}}$ closed under μ

Call-by-name evaluation: $\to^*_{\mathtt{n}}$

Standardization for cbn

Standard reducibility $M \Rightarrow_{\mathtt{n}} N$:

$$\frac{}{x \Rightarrow_{\mathtt{n}} x}\ VAR \quad \frac{M \Rightarrow_{\mathtt{n}} N}{\lambda x.M \Rightarrow_{\mathtt{n}} \lambda x.N}\ ABS \quad \frac{M \Rightarrow_{\mathtt{n}} M' \quad N \Rightarrow_{\mathtt{n}} N'}{MN \Rightarrow_{\mathtt{n}} M'N'}\ APL$$

$$\frac{M \to_{\mathtt{n}}^{*} \lambda x.M' \quad [N/x]M' \Rightarrow_{\mathtt{n}} P}{MN \Rightarrow_{\mathtt{n}} P}\ RDX$$

Standardization theorem: $M \to_{\beta_{\mathtt{n}}}^{*} N$ iff $M \Rightarrow_{\mathtt{n}} N$

Call-by-value: λ_{v}

$$(\mathit{Terms}) \quad M, N ::= V \mid MN \qquad\qquad (\mathit{Values}) \quad V ::= x \mid \lambda x.M$$

$$(\beta_{\mathrm{v}}) \qquad (\lambda x.M)V \to [V/x]M$$

$$\frac{M \to M'}{MN \to M'N}\ (\mu) \qquad \frac{N \to N'}{VN \to VN'}\ (\nu_{val}) \qquad \frac{M \to M'}{\lambda x.M \to \lambda x.M'}\ (\xi)$$

Call-by-value reduction and equality: $\to^*_{\beta_{\mathrm{v}}}$ and $=_{\beta_{\mathrm{v}}}$

$\to_{\mathrm{v}}$: β_{v} closed under μ and ν_{val}

Call-by-value evaluation: $\to^*_{\mathrm{v}}$

Standardization for cbv

Standard reducibility $M \Rightarrow_{\mathrm{v}} N$:

$$\frac{}{x \Rightarrow_{\mathrm{v}} x}\ VAR \qquad \frac{M \Rightarrow_{\mathrm{v}} N}{\lambda x.M \Rightarrow_{\mathrm{v}} \lambda x.N}\ ABS \qquad \frac{M \Rightarrow_{\mathrm{v}} M' \quad N \Rightarrow_{\mathrm{v}} N'}{MN \Rightarrow_{\mathrm{v}} M'N'}\ APL$$

$$\frac{M \rightarrow_{\mathrm{v}}^{*} \lambda x.M' \quad N \rightarrow_{\mathrm{v}}^{*} V \quad [V/x]M' \Rightarrow_{\mathrm{v}} P}{MN \Rightarrow_{\mathrm{v}} P}\ RDX$$

Standardization theorem: $M \rightarrow_{\beta_{\mathrm{v}}}^{*} N$ iff $M \Rightarrow_{\mathrm{v}} N$

MODAL TARGET

AND

MODAL EMBEDDINGS

Modal target: the box calculus λ_{b}

$$M, N ::= \overbrace{\varepsilon(x) \mid \lambda x.M}^{V} \mid MN \mid \underbrace{\mathsf{box}(M)}_{\mathcal{B}}$$

$$(\beta_{\mathsf{b}}) \quad (\lambda x.M)\mathsf{box}(N) \quad \to \quad [N/\varepsilon(x)]M$$

Modal target: the box calculus λ_{b}

$$M, N ::= \overbrace{\varepsilon(x) \mid \lambda x.M}^{V} \mid MN \mid \underbrace{\mathrm{box}(M)}_{\mathcal{B}}$$

$$(\beta_{\mathtt{b}}) \quad (\lambda x.M)\mathrm{box}(N) \quad \rightarrow \quad [N/\varepsilon(x)]M$$

- Call-by-box reduction: $\rightarrow^*_{\beta_{\mathtt{b}}}$
- $\rightarrow_{\mathtt{we}}$: forbid reduction under λ-abstraction and boxes
- Weak and external reduction, or cbb evaluation: $\rightarrow^*_{\mathtt{we}}$

Modal target: the box calculus λ_b

$$A ::= X \mid B \supset A \mid B \qquad\qquad B ::= \Box A$$

Contexts Γ are sets of declarations $x : B$

$$\frac{}{\Gamma, x : \Box A \vdash \varepsilon(x) : A} \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{box}(M) : \Box A}$$

$$\frac{\Gamma, x : B \vdash M : A}{\Gamma \vdash \lambda x.M : B \supset A} \qquad \frac{\Gamma \vdash M : B \supset A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}$$

Calling paradigm cbb

Standard reducibility $M \Rightarrow_{\mathtt{b}} N$:

$$\frac{}{\varepsilon(x) \Rightarrow_{\mathtt{b}} \varepsilon(x)}\ \mathit{VAR} \qquad \frac{M \Rightarrow_{\mathtt{b}} N}{\lambda x.M \Rightarrow_{\mathtt{b}} \lambda x.N}\ \mathit{ABS}$$

$$\frac{M \Rightarrow_{\mathtt{b}} M' \quad N \Rightarrow_{\mathtt{b}} N'}{MN \Rightarrow_{\mathtt{b}} M'N'}\ \mathit{APL} \qquad \frac{M \Rightarrow_{\mathtt{b}} N}{\mathrm{box}(M) \Rightarrow_{\mathtt{b}} \mathrm{box}(N)}\ \mathit{BOX}$$

$$\frac{M \rightarrow^{*}_{\mathtt{we}} \lambda x.M' \quad N \rightarrow^{*}_{\mathtt{we}} \mathrm{box}(N') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathtt{b}} P}{MN \Rightarrow_{\mathtt{b}} P}\ \mathit{RDX}$$

Calling paradigm cbb

Standard reducibility $M \Rightarrow_{\mathtt{b}} N$:

$$\frac{}{\varepsilon(x) \Rightarrow_{\mathtt{b}} \varepsilon(x)}\ VAR \quad \frac{M \Rightarrow_{\mathtt{b}} N}{\lambda x.M \Rightarrow_{\mathtt{b}} \lambda x.N}\ ABS$$

$$\frac{M \Rightarrow_{\mathtt{b}} M' \quad N \Rightarrow_{\mathtt{b}} N'}{MN \Rightarrow_{\mathtt{b}} M'N'}\ APL \quad \frac{M \Rightarrow_{\mathtt{b}} N}{\mathrm{box}(M) \Rightarrow_{\mathtt{b}} \mathrm{box}(N)}\ BOX$$

$$\frac{M \rightarrow^{*}_{\mathtt{we}} \lambda x.M' \quad N \rightarrow^{*}_{\mathtt{we}} \mathrm{box}(N') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathtt{b}} P}{MN \Rightarrow_{\mathtt{b}} P}\ RDX$$

Theorem (Standardization for $\lambda_{\mathtt{b}}$)

$M \rightarrow^{*}_{\beta_{\mathtt{b}}} N$ *iff* $M \Rightarrow_{\mathtt{b}} N$

Girard's embedding: $\lambda_n \to \lambda_b$

Translation of formulas

$$\begin{array}{rcl} X^\circ & = & X \\ (A_1 \supset A_2)^\circ & = & \Box A_1^\circ \supset A_2^\circ \end{array}$$

Translation of terms

$$\begin{array}{rcl} x^\circ & = & \varepsilon(x) \\ (\lambda x.M)^\circ & = & \lambda x.M^\circ \\ (MN)^\circ & = & M^\circ \mathrm{box}(N^\circ) \end{array}$$

Typing

$$\Gamma \vdash M : A \text{ in } \lambda_n \text{ iff } \Box\Gamma^\circ \vdash M^\circ : A^\circ \text{ in } \lambda_b$$

Girard's embedding: $\lambda_{\mathtt{n}} \to \lambda_{\mathtt{b}}$

Theorem (Properties of Girard's translation)

1. *(Preservation and reflection of reduction)* $M \to_{\beta_{\mathtt{n}}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^\circ \to_{\beta_{\mathtt{b}}} N^\circ$ *in* $\lambda_{\mathtt{b}}$*.*
2. *(Preservation and reflection of evaluation)* $M \to_{\mathtt{n}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^\circ \to_{\mathtt{we}} N^\circ$ *in* $\lambda_{\mathtt{b}}$*.*
3. *(Preservation and reflection of standard reduction)* $M \Rightarrow_{\mathtt{n}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^\circ \Rightarrow_{\mathtt{b}} N^\circ$ *in* $\lambda_{\mathtt{b}}$*.*

Corollary

Standardization for $\lambda_{\mathtt{n}}$*.*

Gödel's embedding: $\lambda_{\mathrm{v}} \to \lambda_{\mathrm{b}}$

Translation of formulas

$$A^* = \Box A^\bullet$$

$$\begin{aligned} X^\bullet &= X \\ (A_1 \supset A_2)^\bullet &= \Box A_1^\bullet \supset \Box A_2^\bullet \end{aligned}$$

Translation of terms

$$\begin{aligned} V^* &= \mathsf{box}(V^\bullet) \\ (MN)^* &= \mathsf{raise}(N^*)M^* \end{aligned}$$

$$\begin{aligned} x^\bullet &= \varepsilon(x) \\ (\lambda x.M)^\bullet &= \lambda x.M^* \end{aligned}$$

where

$$\mathsf{raise}(N) := \lambda z.\varepsilon(z)N$$

Gödel's embedding: $\lambda_{\mathtt{v}} \to \lambda_{\mathtt{b}}$

Typing

- $\Gamma \vdash M : A$ in $\lambda_{\mathtt{v}}$ iff $\Gamma^* \vdash M^* : A^*$ in $\lambda_{\mathtt{b}}$
- $\Gamma \vdash V : A$ in $\lambda_{\mathtt{v}}$ iff $\Gamma^* \vdash V^\bullet : A^\bullet$ in $\lambda_{\mathtt{b}}$

Notice the derived typing rule

$$\frac{\Gamma \vdash N : B}{\Gamma \vdash \mathrm{raise}(N) : (\Box(B \supset B')) \supset B'}$$

Gödel's embedding: $\lambda_{\mathtt{v}} \to \lambda_{\mathtt{b}}$

Theorem (Properties of Gödel's translation)

1. *(Preservation and reflection of reduction)* $M \to_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to_{\beta_{\mathtt{b2}}} N^*$ *in* $\lambda_{\mathtt{b}}$.
2. *(Preservation and reflection of evaluation)* $M \to_{\mathtt{v}}^* V$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to_{\mathtt{we}}^* V^*$ *in* $\lambda_{\mathtt{b}}$.
3. *(Preservation and reflection of standard red.)* $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \Rightarrow_{\mathtt{b2}} N^*$ *in* $\lambda_{\mathtt{b}}$.

Gödel's embedding: $\lambda_{\mathtt{v}} \to \lambda_{\mathtt{b}}$

Theorem (Properties of Gödel's translation)

1. *(Preservation and reflection of reduction)* $M \to_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to_{\beta_{\mathtt{b2}}} N^*$ *in* $\lambda_{\mathtt{b}}$.
2. *(Preservation and reflection of evaluation)* $M \to^*_{\mathtt{v}} V$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to^*_{\mathtt{we}} V^*$ *in* $\lambda_{\mathtt{b}}$.
3. *(Preservation and reflection of standard red.)* $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \Rightarrow_{\mathtt{b2}} N^*$ *in* $\lambda_{\mathtt{b}}$.

Corollary

Standardization for $\lambda_{\mathtt{v}}$.

The extra bits in the cbv case

In $\lambda_{\mathtt{b}}$ define $\beta_{\mathtt{b2}} \subset \to^2_{\mathtt{we}}$:

$$\mathrm{raise}(\mathrm{box}(N))\mathrm{box}(\lambda x.P) \to [N/\varepsilon(x)]P$$

In $\lambda_{\mathtt{b}}$ define $\Rightarrow_{\mathtt{b2}} \subset \Rightarrow_{\mathtt{b}}$, by replacing

$$\frac{M \to^*_{\mathtt{we}} \lambda x.M' \quad N \to^*_{\mathtt{we}} \mathrm{box}(N') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathtt{b}} P}{MN \Rightarrow_{\mathtt{b}} P}\ RDX$$

with

$$\frac{N \to^*_{\mathtt{we}} \mathrm{box}(N') \quad M \to^*_{\mathtt{we}} \mathrm{box}(\lambda x.M') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathtt{b2}} Q}{\mathrm{raise}(N)M \Rightarrow_{\mathtt{b2}} Q}\ RDX2$$

First narrative[1]

- The modal target is a new calling paradigm, call-by-box (cbb)
- The modal embeddings implement "protect-by-a-box", an abstract form of the compilation technique "protect-by-a-lambda"
- "Protect-by-a-box" has a cbn side and a cbv side
- The modal embeddings unify cbn and cbv inside cbb

[1] JES, L. Pinto, T. Uustalu, "Modal embeddings and calling paradigms", FSCD 2019

MORE ONE EMBEDDINGS AND PARADIGMS

Needed

- Improve the properties of Gödel's embedding
- Change the narrative, to fit better with the strong properties of Girard's embedding
- Separation of the extra bits by modal reasons

Redesign the modal target

Starting from λ_{b}

- Forbid types of the form $\Box\Box A$
- Separate, already in the untyped syntax, terms that must have a modal type from those that cannot have a modal type
- Application will remain an ambiguous constructor: separate the two forms (and the corresponding reduction rules)
- Just keep the particular forms of application that show up in the image of the embeddings

Types, terms and reduction

$$
\begin{array}{rlll}
\text{(Types)} & A & ::= & B \mid C \\
\text{(Boxed types)} & B & ::= & \Box C \\
\text{(Unboxed types)} & C & ::= & X \mid B \supset A
\end{array}
$$

Types, terms and reduction

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\end{array}
$$

$$
\begin{array}{rrcl}
\text{(Terms)} & T & ::= & M \mid P \\
\text{(Unboxed terms)} & M, N & ::= & \overbrace{\varepsilon(x) \mid \lambda x.T}^{V} \mid M @_{\not{\mathfrak{b}}} Q \\
\text{(Boxed terms)} & P, Q & ::= & \underbrace{\mathsf{box}(M)}_{\mathcal{B}} \mid M @_{\mathfrak{b}} Q
\end{array}
$$

Types, terms and reduction

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$$(\lambda x.T) @ \mathsf{box}(N) \to [N/\varepsilon(x)]M$$

Derived notion of application and reduction

$$\dfrac{\dfrac{\dfrac{}{\Gamma, z : \Box(B' \supset B) \vdash \varepsilon(z) : B' \supset B} \quad \dfrac{\color{blue}{\Gamma \vdash Q : B'}}{\Gamma, z : \Box(B' \supset B) \vdash Q : B'}\; W}{\dfrac{\Gamma, z : \Box(B' \supset B) \vdash \varepsilon(z) @_{\mathtt{b}} Q : B}{\Gamma \vdash \lambda z.\varepsilon(z) @_{\mathtt{b}} Q : (\Box(B' \supset B)) \supset B}} \qquad \color{blue}{\Gamma \vdash P : \Box(B' \supset B)}}{\Gamma \vdash \underbrace{(\lambda z.\varepsilon(z) @_{\mathtt{b}} Q) @_{\mathtt{b}} P}_{\color{blue}{QP}} : B}$$

Derived notion of application and reduction

$$\dfrac{\dfrac{\dfrac{}{\Gamma, z : \Box(B' \supset B) \vdash \varepsilon(z) : B' \supset B} \quad \dfrac{\Gamma \vdash Q : B'}{\Gamma, z : \Box(B' \supset B) \vdash Q : B'}\ W}{\dfrac{\Gamma, z : \Box(B' \supset B) \vdash \varepsilon(z) @_{\mathtt{b}} Q : B}{\Gamma \vdash \lambda z.\varepsilon(z) @_{\mathtt{b}} Q : (\Box(B' \supset B)) \supset B}} \qquad \Gamma \vdash P : \Box(B' \supset B)}{\Gamma \vdash \underbrace{(\lambda z.\varepsilon(z) @_{\mathtt{b}} Q) @_{\mathtt{b}} P}_{QP} : B}$$

$$\mathrm{box}(N)(\mathrm{box}(\lambda x.P)) \to [N/\varepsilon(x)]P$$

The final syntax

$$
\begin{array}{rrcl}
\text{(Terms)} & T & ::= & M \mid P \\[2em]
\text{(Unboxed terms)} & M, N & ::= & \overbrace{\varepsilon(x) \mid \lambda x.\, T}^{V} \mid M @_{\not{\mathfrak{b}}} Q \\
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\end{array}
$$

The general form of (un)boxed terms

$$
VQ_1 \cdots Q_m \qquad\qquad Q_m \cdots Q_1\mathcal{B}
$$

The final syntax: another box calculus $\lambda_{\between}$

$$
\begin{array}{lrcl}
\text{(Terms)} & T & ::= & M \mid P
\end{array}
$$

$$
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\end{array}
$$

The general form of (un)boxed terms

$$
VQ_1 \cdots Q_m \qquad\qquad Q_m \cdots Q_1\mathcal{B}
$$

Two reduction rules

$$
\begin{array}{lrcl}
(\beta_<) & (\lambda x.M)\text{box}(N) & \to & [N/\varepsilon(x)]M \\
(\beta_>) & \text{box}(N)(\text{box}(\lambda x.P)) & \to & [N/\varepsilon(x)]P
\end{array}
$$

Girard's translation: choose the unboxed mode

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\end{array}
$$

Girard's translation: choose the unboxed mode

$$\text{(Terms)} \quad T ::= M \mid {\color{red}P}$$

$$\begin{array}{rrcl}
\text{(Unboxed terms)} & M, N & ::= & \overbrace{\varepsilon(x) \mid \lambda x.T}^{V} \mid MQ \\
\text{(Boxed terms)} & P, Q & ::= & \underbrace{\mathrm{box}(M)}_{\mathcal{B}} \mid {\color{red}QP}
\end{array}$$

The general form of (un)boxed terms

$$VQ_1 \cdots Q_m \qquad\qquad Q_m \cdots Q_1\mathcal{B}$$

Two reduction rules

$$\begin{array}{lrcl}
(\beta_<) & (\lambda x.M)\mathrm{box}(N) & \to & [N/\varepsilon(x)]M \\
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Girard's translation: choose the unboxed mode

$$\text{(Terms)} \quad T ::= M \mid {\color{red}P}$$

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\text{(Unboxed terms)} & M, N & ::= & \overbrace{\varepsilon(x) \mid \lambda x.{\color{blue}M}}^{V} \mid MQ \\
\text{(Boxed terms)} & P, Q & ::= & \underbrace{\text{box}(M)}_{\mathcal{B}} \mid {\color{red}QP}
\end{array}$$

The general form of (un)boxed terms

$$VQ_1 \cdots Q_m \qquad\qquad Q_m \cdots Q_1\mathcal{B}$$

Two reduction rules

$$\begin{array}{lrcl}
(\beta_<) & (\lambda x.M)\text{box}(N) & \to & [N/\varepsilon(x)]M \\
(\beta_>) & \text{box}(N)(\text{box}(\lambda x.P)) & \to & [N/\varepsilon(x)]P
\end{array}$$

Girard's translation: choose the unboxed mode

$$
\begin{array}{lrcl}
\text{(Terms)} & T & ::= & M \mid \color{red}{P} \\
\text{(Unboxed terms)} & M, N & ::= & \overbrace{\varepsilon(x) \mid \lambda x.\color{blue}{M}}^{V} \mid M\color{blue}{\text{box}(N)} \\
\text{(Boxed terms)} & P, Q & ::= & \underbrace{\text{box}(M)}_{\mathcal{B}} \mid \color{red}{QP}
\end{array}
$$

The general form of (un)boxed terms

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VQ_1 \cdots Q_m \qquad\qquad Q_m \cdots Q_1\mathcal{B}
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$$\text{(Terms)} \qquad T \quad ::= \quad M \mid P$$

$$\begin{array}{rrcl} \text{(Unboxed terms)} & M, N & ::= & \overbrace{\varepsilon(x) \mid \lambda x.M}^{V} \mid M\mathrm{box}(N) \\ \text{(Boxed terms)} & P, Q & ::= & \underbrace{\mathrm{box}(M)}_{\mathcal{B}} \mid QP \end{array}$$

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The general form of (un)boxed terms

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VQ_1 \cdots Q_m \qquad {\color{red}Q_m \cdots Q_1 \mathcal{B}}
$$

Two reduction rules

$$
\begin{array}{lrcl}
(\beta_<) & (\lambda x.M)\mathrm{box}(N) & \to & [N/\varepsilon(x)]M \\
{\color{red}(\beta_>)} & {\color{red}\mathrm{box}(N)(\mathrm{box}(\lambda x.P))} & {\color{red}\to} & {\color{red}[N/\varepsilon(x)]P}
\end{array}
$$

Gödel's translation: choose the boxed mode

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$$
\begin{array}{rrcl}
\text{(Terms)} & T & ::= & M \mid P \\
\text{(Unboxed terms)} & M, N & ::= & \overbrace{\varepsilon(x) \mid \lambda x.T}^{V} \mid MQ \\
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The general form of (un)boxed terms

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\begin{array}{lrcl}
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\begin{array}{rrcl}
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Gödel's translation: choose the boxed mode

$$\begin{array}{rrcl} \text{(Terms)} & T & ::= & M \mid P \\ \\ \text{(Unboxed terms)} & V & ::= & \overbrace{\varepsilon(x) \mid \lambda x.P}^{\mathcal{V}} \mid MQ \\ \text{(Boxed terms)} & P, Q & ::= & \underbrace{\text{box}(V)}_{\mathcal{B}} \mid QP \end{array}$$

The general form of (un)boxed terms

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Gödel's translation: choose the boxed mode

$$\text{(Terms)} \quad T ::= M \mid P$$

$$\begin{array}{rrcl} \text{(Unboxed terms)} & V & ::= & \overbrace{\varepsilon(x) \mid \lambda x.P}^{V} \mid MQ \\ \text{(Boxed terms)} & P, Q & ::= & \underbrace{\mathrm{box}(V)}_{\mathcal{B}} \mid QP \end{array}$$

The general form of (un)boxed terms

$$VQ_1 \cdots Q_m \qquad\qquad Q_m \cdots Q_1 \mathcal{B}$$

Two reduction rules

$$\begin{array}{lrcl} (\beta_<) & (\lambda x.M)\mathrm{box}(N) & \to & [N/\varepsilon(x)]M \\ (\beta_>) & \mathrm{box}(N)(\mathrm{box}(\lambda x.P)) & \to & [N/\varepsilon(x)]P \end{array}$$

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The general form of (un)boxed terms

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Two reduction rules

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\begin{array}{rrcl}
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(\beta_>) & \text{box}(N)(\text{box}(\lambda x.P)) & \to & [N/\varepsilon(x)]P
\end{array}
$$

Results

Theorem (Properties of Girard's translation from $\lambda_{\mathtt{n}}$ to $\lambda_{\between}$)

1. $M \rightarrow_{\beta_{\mathtt{n}}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^{\circ} \rightarrow_{\beta_{<}} N^{\circ}$ *in* $\lambda_{\between}$.
2. $M \rightarrow^{*}_{\mathtt{n}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^{\circ} \twoheadrightarrow_{\mathtt{b}} N^{\circ}$ *in* $\lambda_{\between}$.
3. $M \Rightarrow_{\mathtt{n}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^{\circ} \Rightarrow_{\between} N^{\circ}$ *in* $\lambda_{\between}$.

Theorem (Properties of Gödel's translation from $\lambda_{\mathtt{v}}$ to $\lambda_{\between}$)

1. $M \rightarrow_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^{*} \rightarrow_{\beta_{>}} N^{*}$ *in* $\lambda_{\between}$.
2. $M \rightarrow^{*}_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^{*} \twoheadrightarrow_{\mathtt{b}} N^{*}$ *in* $\lambda_{\between}$.
3. $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^{*} \Rightarrow_{\between} N^{*}$ *in* $\lambda_{\mathtt{b}}$.

Second narrative[2]

- The modal target still follows call-by-box and combines two "modes" of the application constructor and reduction
- Each mode corresponds to a calling paradigm cbn or cbv
- The modal embeddings point out isomorphic copies of cbn or cbn inside cbb
- Cbn and Cbv coexist inside cbb

[2] JES, L. Pinto, T. Uustalu, "Plotkin's lambda-calculus as a modal calculus", JLAMP, 2022

Work in progress

- Explore the redesigned modal calculus $\lambda_{\between}$
- Work out fully the several instantiations

THANK YOU