Canonicity and Decidability of Equality for Setoid Type Theory

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To express algebraic theories with equalities.

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Quotient Inductive Types

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How?

Using Setoid Type Theory

Examples

Functional extensionality

- $C = A \to B$ $(A, B \in Set)$
- $f \sim g = \forall a \in A : f(a) = g(a)$

Propositional extensionality

- $C = Prop$
- $P \sim Q = (P \to Q) \times (Q \to P)$

Semantical program equivalence

- C : Syntax of programming language
- $\_ \sim \_$: Reflexive, symmetric and transitive closure of semantic transitions

Also: Bisimilarity of coinductive datatype instances

Example - Teaching

Type System course

- Algebraic models of programming languages
- Well typed syntax with equations

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Problem

Agda inefficiency

- Manual transports
- Rewrite rules

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Aim

Better computability

- No additional axioms necessary
- Avoiding transport hell

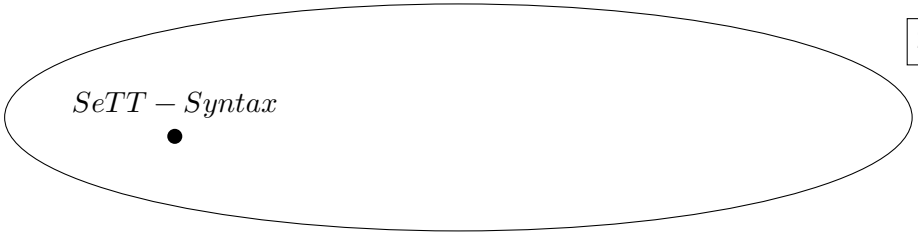
Setoid Type Theory

We need canonicity.

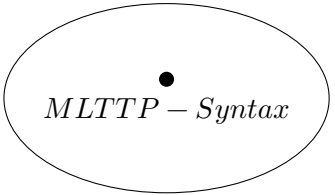
Previously:

- Observational Equality, Now! - Altenkirch et al. [2007]
- Cubical Syntax for Reflection-Free Extensional Equality - Sterling et al. [2019]
- Shallow Embedding of Type Theory is Morally Correct - Kaposi et al. [2019]
- Setoid type theory - a syntactic translation - Altenkirch et al. [2019]
- Constructing a universe for the setoid model - Altenkirch et al. [2021]
- Observational Equality: Now For Good - Pujet et al. [2021]

Overview



Setoid Type Theory



Martin-Löf Type Theory

Decidable Equality
Canonicity

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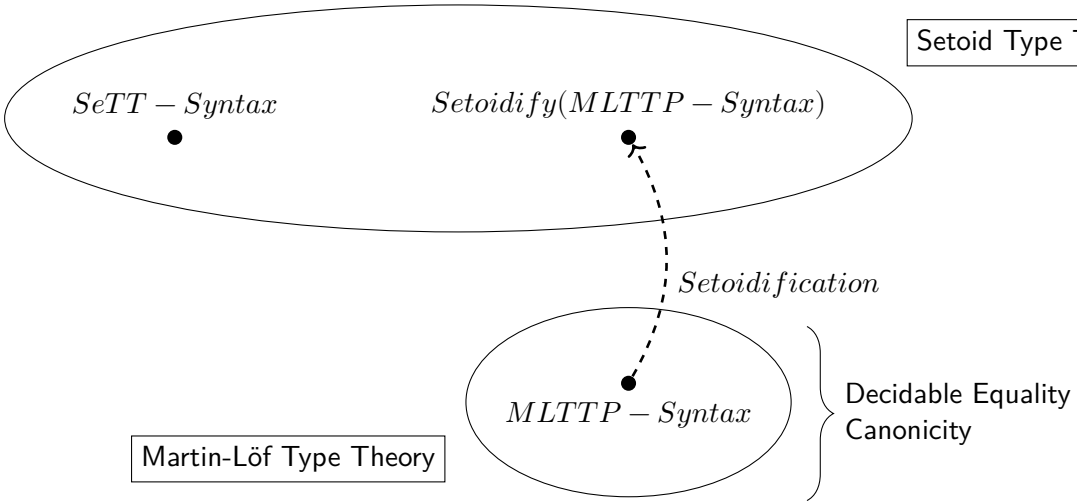
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$Setoidify(MLTTP - Syntax)$
$Setoidification$
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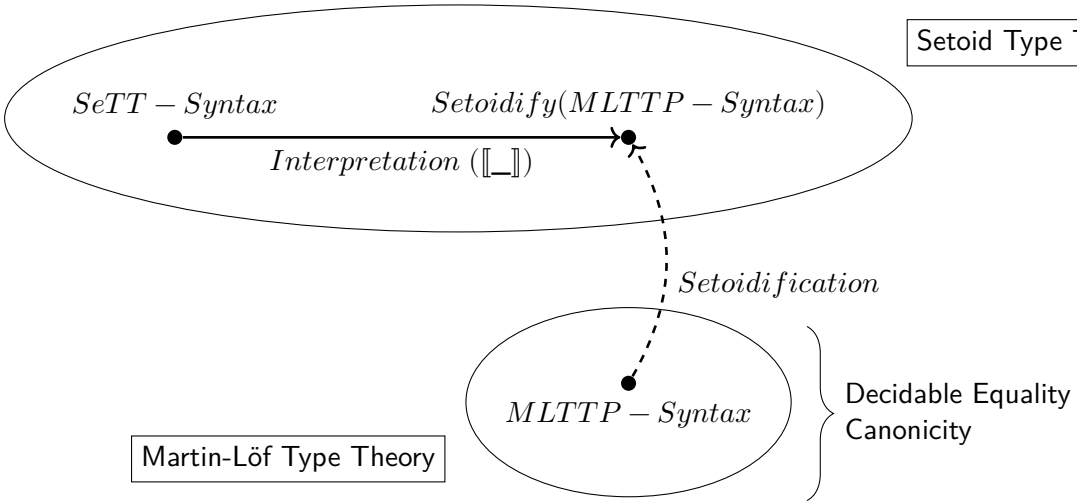
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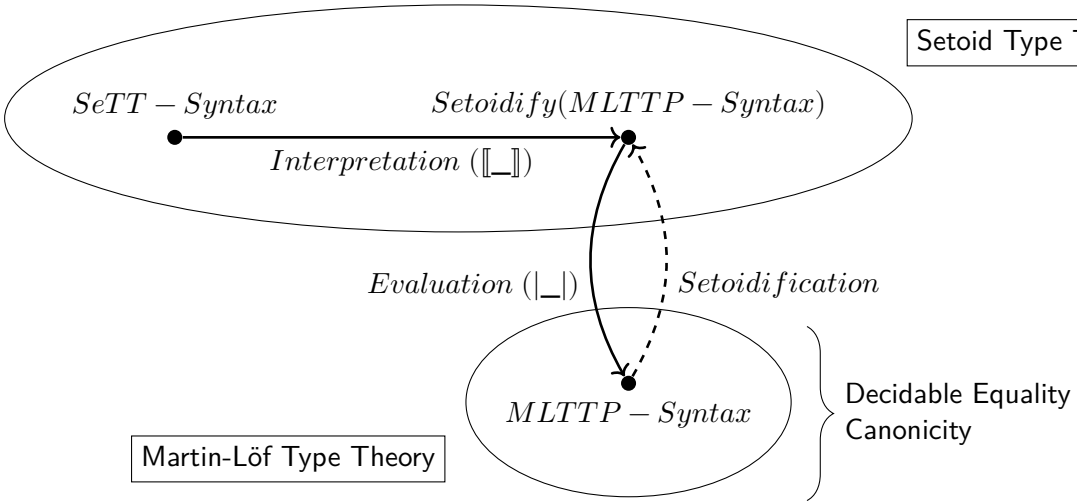
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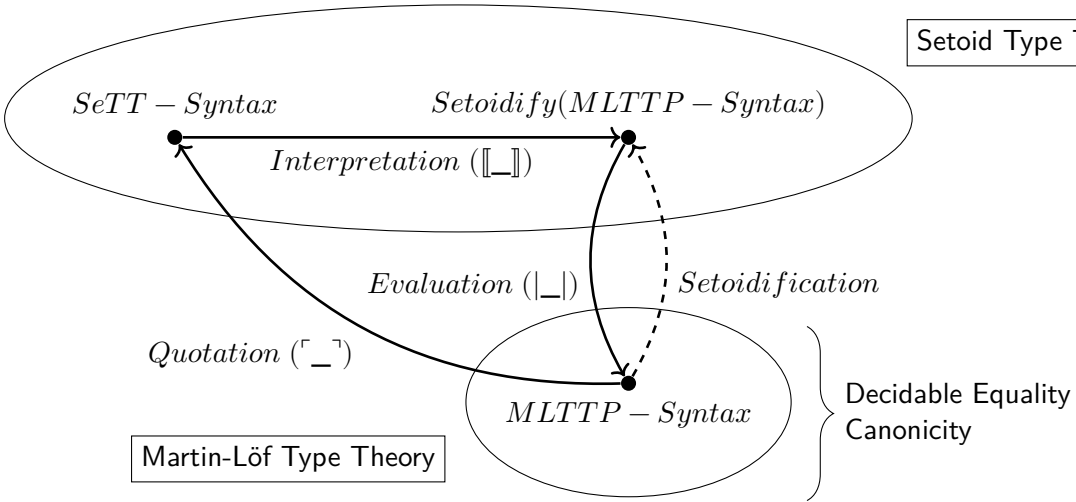
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Constructing a setoid model from an MLTTP model.

Types $(A : Setoidify(MLTTP\ model).Ty)$

- $|A|$: type from the input MLTTP model
- $A \sim$: heterogeneous equivalence relation over terms of the type
- RA, SA, TA : the relation is reflexive, symmetric and transitive
- $coeA$: coercion over context equivalences
- $cohA$: the result of coercion is in relation with the original

Properties

Let the combined $|[\![\_]\!]|$ operation be the interpretation and evaluation from the syntax of SeTT to the setoidified MLTTP syntax.
The injectivity of this composite function: $\underline{|[\![t]\!]| = |[\![t']\!]| \to t = t'}$ where

- t, t' are terms in the syntax of SeTT
- $|[\![t]\!]|, |[\![t']\!]|$ are terms in the syntax of MLTTP

Decidability of equality

- $|[\![t]\!]| = |[\![t']\!]|$ implies $t = t'$ by injectivity
- $|[\![t]\!]| \neq |[\![t']\!]|$ implies $t \neq t'$ by contradiction

Canonicity

For example, on type Bool:

- $|[\![t]\!]| = M.false$ implies $t = S.false$ by injectivity
- $|[\![t]\!]| = M.true$ implies $t = S.true$ by injectivity

Injectivity

Dependent model, where:

- $Con^\bullet\ \Gamma :=$ comparison map from Γ to $\ulcorner|[\![\Gamma]\!]|\urcorner$
- $Sub^\bullet\ \Delta^\bullet\ \Gamma^\bullet\ \gamma := (\Gamma^\bullet \circ \gamma = \ulcorner|[\![\gamma]\!]|\urcorner \circ \Delta^\bullet)$
- $Ty^\bullet\ \Gamma^\bullet\ A :=$ isomorphism between terms of type A and $\ulcorner|[\![A]\!]|\urcorner$
- $Tm^\bullet\ \Gamma^\bullet\ A^\bullet\ t := (A^\bullet[id, t] = \ulcorner|[\![t]\!]|\urcorner[\Gamma^\bullet])$

Injectivity from the dependent model:

- Given: $[\![t]\!] = [\![t']\!]$
- We have: $A^\bullet[id,\ t\ ] = \ulcorner|[\![t]\!]|\urcorner\ [\Gamma^\bullet] = \ulcorner|[\![t']\!]|\urcorner\ [\Gamma^\bullet] = A^\bullet[id,\ t'\ ]$

Issues

Eliminator terms for types without η rules applied on terms with context-dependent types.
Example: Bool + Sigma

$$\Sigma^{\bullet}[id, ite\ u\ v\ t] = ... = (\_, \_) \neq ite\ \_\ \_\ t = ... = \ulcorner[\![|ite\ u\ v\ t|]\!]\urcorner[\Gamma^{\bullet}]$$

Possible solutions:

- Add η rules?
- Prove the equalities type by type?

Thank you for your attention!

Questions?

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