Deciding inclusion in cofibration classifiers

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Motivation

Cubical type theories[1,2,3] were invented to realize the potential of intensional Martin-Löf type theory to become a computational theory of *homotopy types* (or even *directed* homotopy types).

The standard semantics for cubical type theories are cubical presheaves, which serve as combinatorial models of (directed) homotopy theory. Witnesses of equality types are implemented as paths out of a special interval type, whose standard semantics is the 1-dimensional cubical presheaf $\square^1$:

$$0 \xrightarrow{x_1} 1$$

[1] Cyril Cohen et al. *Cubical Type Theory: a constructive interpretation of the univalence axiom*. 2016. arXiv: 1611.02108 [cs.LO].

[2] Carlo Angiuli et al. “Syntax and models of Cartesian cubical type theory”. In: *Mathematical Structures in Computer Science* 31.4 (2021), pp. 424–468.

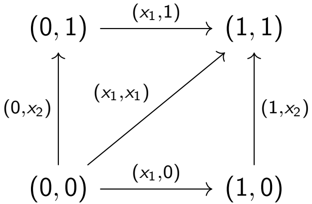
[3] Matthew Z. Weaver and Daniel R. Licata. “A Constructive Model of Directed Univalence in Bicubical Sets”. In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. Saarbrücken, Germany: Association for Computing Machinery, 2020, pp. 915–928.

Morphisms in $\square^n$

This talk will be based on the following definition of $\square^n$:

$$\square^n_m ::= \left\{ (h_1, \ldots, h_n) \mid h_i \in \text{FreeDist}\left(\{x_1, \ldots, x_m\}\right) \right\}$$

For example, the non-degenerate 0- and 1-cubes in $\square^2$ may be pictured as

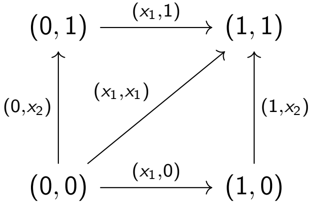


Cofibration classifier

Cartesian cubical type theories include an inductive description of a special collection of cubical subsets, a *cofibration classifier* Ψ, obtained by combining non-degenerate cubes in $\square^n$, using binary union and intersection. Along with $\emptyset$ and $\square^n$, the following generators are shared among these theories:

$$(x_i = 1)^n_m ::= \{h : [m] \to [n] \mid h_i = 1\}$$
$$(x_i = 0)^n_m ::= \{h : [m] \to [n] \mid h_i = 0\}$$

Cofibration classifier



For example, the upper left corner point can be specified as

$$(x_1 = 0)^2 \cap (x_2 = 1)^2$$

The boundary of the square can be specified as

$$(x_1 = 0)^2 \cup (x_1 = 1)^2 \cup (x_2 = 0)^2 \cup (x_2 = 1)^2$$

Cofibration classifier

In this talk, an element $\psi^n \in \Psi_n$ is defined by:

$$\begin{aligned} \phi^n, \chi^n, \psi^n &::= \sigma^n \mid \sigma^n \to \sigma^n \mid \phi^n \cap \phi^n \mid \phi^n \cup \phi^n \\ \sigma^n, \tau^n &::= \emptyset \mid \Box^n \mid (x_i = 1)^n \mid \sigma^n \cap \sigma^n \mid \sigma^n \cup \sigma^n \end{aligned}$$

Note that

$$\big((x_i = 1)^n \to \emptyset\big) \equiv (x_i = 0)^n$$

Decision problem

Given cubical subsets $\phi^n, \psi^n \in \Psi_n$, is it the case that $\phi^n \subseteq \psi^n$?

Corner points

Consider the discrete presheaf containing the corner points of the *n*-cube:

$$B^n_m ::= \{0,1\}^n$$

Obviously, if $\phi^n \subseteq \psi^n$, then $\phi^n \cap B^n \subseteq \psi^n \cap B^n$. Now if the converse were true, then we would have an almost trivial reduction to coSAT, since for any $b \in B^n_0$, $\{b\} \subseteq \phi^n$ can be decided by checking whether ϕ^n "evaluated" at b is $\emptyset$ or $\Box^n$.

Of course, if the converse were true, it would follow immediately that $\Box^1 = B^1$. However, it turns out that a less ridiculous proposition holds which is just as good.

A transformation F

Without defining it explicitly, let's say that there is a transformation F of ψ^n, parameterized by ϕ^n, into a cubical subset of a cube of a not much higher dimension than n; and that F has the following virtue:

Theorem

Let $\phi^n, \psi^n \in \Psi_n$. Let $p = kn$ where k is the number of implications in ψ_n. Then

$$\phi^n \subseteq \psi^n \qquad \textit{iff} \qquad \phi^p \cap B^p \subseteq F(\phi^n, \psi^n) \cap B^p$$

where $\phi^p = y(x_1, \ldots, x_n)^* \phi^n$.

Example: invalidity of LEM

Let $\phi^1 = \Box^1$ and let $\psi^1 = \left((x_1 = 1)^1 \to \emptyset\right) \cup (x_1 = 1)^1$. Is it the case that $\phi^1 \subseteq \psi^1$?

To apply the theorem, first we expand and simplify (for human readers)

$$\begin{aligned} F(\Box^1, \psi^1) &= \Big(\big((x_1 = 1)^2 \to (x_2 = 1)^2\big) \cap \Box^2 \cap (x_2 = 1)^2 \to \emptyset\Big) \cup (x_1 = 1)^2 \\ &= \big((x_2 = 1)^2 \to \emptyset\big) \cup (x_1 = 1)^2 \end{aligned}$$

Hence

$$\begin{aligned} \Box^2 \cap B^2 &\subseteq F(\Box^1, \psi^1) \cap B^2 \\ B^2 &\subseteq \Big(\big((x_2 = 1)^2 \to \emptyset\big) \cup (x_1 = 1)^2\Big) \cap B^2 \\ B_0^2 &\subseteq \big\{(0,0), (1,0), (1,1)\big\} \end{aligned}$$

which does not hold, as desired.

Example: classical and intuitionistic implication

Let $\phi^2 = \big((x_1 = 1)^2 \to \emptyset\big) \cup (x_2 = 1)^2$ and let $\psi^2 = (x_1 = 1)^2 \to (x_2 = 1)^2$. Is it the case that $\phi^2 \subseteq \psi^2$?

We expand and simplify

$$\begin{aligned} F(\phi^2, \psi^2) = \Bigg[& ((x_1 = 1)^4 \to (x_3 = 1)^4) \cap ((x_2 = 1)^4 \to (x_4 = 1)^4) \\ & \cap \Big(\big((x_3 = 1)^4 \to \emptyset\big) \cup (x_4 = 1)^4 \Big) \\ & \cap (x_3 = 1)^4 \Bigg] \to (x_4 = 1)^4 \\ = {} & (x_3 = 1)^4 \cap (x_4 = 1)^4 \to (x_4 = 1)^4 \\ = {} & \square^4 \end{aligned}$$

Hence

$$\begin{aligned} \phi^4 \cap B^4 &\subseteq F(\phi^2, \psi^2) \cap B^4 \\ &\subseteq B^4 \qquad \text{which holds, as desired.} \end{aligned}$$

Example: classical and intuitionistic implication

Let $\phi^2 = (x_1 = 1)^2 \to (x_2 = 1)^2$ and let $\psi^2 = \big((x_1 = 1)^2 \to \emptyset\big) \cup (x_2 = 1)^2$. Is it the case that $\phi^2 \subseteq \psi^2$?

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$$
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&\cap \big((x_3 = 1)^4 \to (x_4 = 1)^4\big) \\
&\cap (x_3 = 1)^4\Bigg] \to \emptyset\Bigg] \cup (x_2 = 1)^4 \\
= \Big(\big((x_3 = 1)^4 &\cap (x_4 = 1)^4\big) \to \emptyset\Big) \cup (x_2 = 1)^4
\end{aligned}
$$

We note that $(0, 0, 1, 1) \in \phi^4$ while

$$
(0, 0, 1, 1) \notin \Big(\big((x_3 = 1)^4 \cap (x_4 = 1)^4\big) \to \emptyset\Big) \cup (x_2 = 1)^4
$$

Definition of F

Let $\phi^n, \psi^n \in \Psi_n$. Compute $p = kn$ where k is the number of implications in ψ^n, which we index starting with $j = 1$. The transformation F is then a cubical subset of $\square^p$ defined inductively on ψ^n:

$$
\begin{aligned}
F(\phi^n, \sigma^n \to_j \tau^n) = \Big[\bigcap_i &\big((x_i = 1) \to (x_{jn+i} = 1)\big) \\
&\cap y(x_{jn+1}, \ldots, x_{(j+1)n})^* \phi^n \\
&\cap y(x_{jn+1}, \ldots, x_{(j+1)n})^* \sigma^n \Big] \\
&\to y(x_{jn+1}, \ldots, x_{(j+1)n})^* \tau^n \\
F(\phi^n, \sigma^n) &= y(x_1, \ldots, x_n)^* \sigma^n \\
F(\phi^n, \chi^n \cap \psi^n) &= F(\phi^n, \chi^n) \cap F(\phi^n, \psi^n) \\
F(\phi^n, \chi^n \cup \psi^n) &= F(\phi^n, \chi^n) \cup F(\phi^n, \psi^n)
\end{aligned}
$$