EQUIVALENCE BETWEEN TYPED AND UNTYPED ALGORITHMIC CONVERSION

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A TALE OF TWO (OR FOUR) CONVERSIONS
Typed and Untyped Conversions, Declarative and Algorithmic

- Two traditions: MLTT (typed) vs PTS (untyped)
  - Typed: good story for η laws
  - Untyped: more efficient, thus used in COQ

Declarative and Algorithmic Conversion

- Declarative: standard presentation, but no direct algorithm
- Algorithmic: easy to relate to an algorithm, but not a good specification
### Typed and Untyped Conversion

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## Typed and Untyped Conversions, Declarative and Algorithmic

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The Situation So Far

Typed declarative (“MLTT”) \[\rightarrow\] [AC07] \[\rightarrow\] [AÖV18] \[\rightarrow\] [SH12] \[\rightarrow\] \[\rightarrow\]

Untyped declarative (“PTS”) \[\leftrightarrow\] \[\leftrightarrow\] \[\leftrightarrow\] \[\leftrightarrow\] Untyped algorithmic (“Coq”)

Typed algorithmic (“Agda”)

• [AC07], [AÖV18]: stronger logical power than the studied system
• [SH12], METACOQ: no η laws
• [AC07], [AÖV18]: stronger logical power than the studied system
• [SH12], METACOQ: no η laws
• !!: Can we do this? With a low logical power?
HOW DO WE DO THIS?
Typed conversion: put bidirectional lenses on

• \( \Gamma \vdash t \leftrightarrow t' : T \) with \( T \) as input, \( \Gamma \vdash n \leftrightarrow n' : T \) with \( T \) as output

• Motto: Conversion \( \leftrightarrow \) checks, neutral comparison \( \leftrightarrow \) infers

\[
\begin{align*}
\Gamma, x : A \vdash f \; x \leftrightarrow g \; x : B \\
\Gamma \vdash f \leftrightarrow g : \Pi \; x : A. \; B
\end{align*}
\]
### Typed conversion: put bidirectional lenses on

- $\Gamma \vdash t \leftrightarrow t' : T$ with $T$ as input, $\Gamma \vdash n \leftrightarrow n' : T$ with $T$ as output.
- Motto: Conversion $\iff$ checks, neutral comparison $\iff$ infers

\[
\frac{\Gamma, x : A \vdash f \ x \iff g \ x : B}{\Gamma \vdash f \iff g : \Pi \ x : A. \ B}
\]

### Untyped conversion

- Same general structure: conversion + neutral comparison.
- Main difference: term-directed instead of type-directed.

\[
\begin{align*}
n \ x & \iff t \quad n \text{ neutral} \\
\frac{n \iff \lambda \ x : A. \ t}{+ \text{ symmetric}}
\end{align*}
\]

\[
\begin{align*}
t & \iff t' \\
\frac{\lambda \ x : A. \ t \iff \lambda \ x : A'. \ t'}{}
\end{align*}
\]
A Proof in Two Steps

Step 1: McBride’s discipline
- Flow of well-formation information for well-behaved bidirectional rules
- Respected by the relation
- Needs meta-theory of the typed variant

Step 2: Relate the rules
- Reasoning on weak-head normal forms
- Rather straightforward

Work in progress, worked out on a toy system ($\lambda \Pi \Box$) on paper.
Does it scale all the way to PCUIC?
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THANK YOU!

