Type Theories with Universe Level Judgments

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Universe polymorphism (Huet 1987, Harper-Pollack 1991). Some starting points for our discussions:

- Escardó’s universe polymorphic development in *Introduction to Univalent Foundations of Mathematics with Agda* 
  www.cs.bham.ac.uk/~mhe/HoTT-UF-in-Agda-Lecture-Notes/

- Voevodsky 2014: *Universe polymorphic type system* - a system with constraints on universe levels. See also Courant’s *EPECC 2002* and Herbelin’s *algebraic universes 2014*.

- An algorithm for solving such constraints (Bezem and Coquand 2021, Bezem, Nieuwenhuis, and Rodriguez 2008: *The Max-Atom Problem and Its Relevance*).

Looking for a simple foundation for universe polymorphism, using ideas from Coq, Agda, and Voevodsky.
The universe polymorphism of univalence

From *Introduction to Univalent Foundations of Mathematics with Agda*:

\[ \text{IdEq} : \{U : \text{Universe}\} \rightarrow (X Y : U') \rightarrow X \equiv Y \rightarrow X \simeq Y \]

\[ \text{IdEq} XX (\text{refl} X) = \text{idstimeq} X \]

\[ \text{isunivalent} : (U : \text{Universe}) \rightarrow U^+ . \]

\[ \text{isunivalent} U = (X Y : U') \rightarrow \text{isequiv} (\text{IdEq} XY) \]
Agda’s system of universes

- An $\omega$-tower of universes à la Russell
  \[ \text{Set}_0 : \text{Set}_1 : \text{Set}_2 : \cdots \]

- A special type \( \text{Level} : \text{Set}_0 \) of universe levels. Universe polymorphism by quantification over \( \text{Level} \).

- Operations \( \text{lzero}, \text{lsuccessor}, \) and \( \text{lsuccessor} \) on levels.

- A "kind" \( \text{Set}_\omega \) which is \textit{not} a type. Moreover,
  \( \text{Set}_\omega \) \textit{is itself the first step in another infinite hierarchy}
  \( \text{Set}_{\omega_i} : \text{Set}_{\omega_{i+1}} \). \textit{However, this hierarchy does not support universe polymorphism, i.e. there are no sorts} \( \text{Set}_\omega l \) \textit{for} \( l : \text{Level} \).
A theory with level judgments

We introduce new (hypothetical) judgment forms

\[ l \text{ Level} \quad l = m \]

Contexts can contain level variables

\[ \alpha \text{ Level} \]

Some rules:

\[
\begin{array}{c}
\frac{\alpha \text{ Level } (\Gamma)}{(\alpha \text{ in } \Gamma)} \\
\frac{l \text{ Level}}{l^+ \text{ Level}} \\
\frac{l \text{ Level} \quad m \text{ Level}}{l \lor m \text{ Level}}
\end{array}
\]

Equations for a $\lor$-semilattice with an inflationary endomorphism $^+$:

\[
\begin{align*}
l \lor l^+ &= l^+ \\
(l \lor m)^+ &= l^+ \lor m^+
\end{align*}
\]

No 0-level.
Rules for level-indexed universes (Tarski-style)

Formation rules

\[
\begin{array}{ll}
\text{l Level} & \text{A : U}_l \\
\text{U}_l \text{ type} & \text{T}_l(A) \text{ type}
\end{array}
\]

Introduction rules

\[
\begin{array}{ll}
\text{A : U}_l & \text{B : T}_l(A) \rightarrow \text{U}_m \\
\Pi^{l,m}AB : \text{U}_{l\wedge m}
\end{array}
\]

Conversion rules

\[
\begin{align*}
\text{T}_{l\wedge m} (\Pi^{l,m}AB) & = \Pi(x : \text{T}_l(A)) \text{T}_m(Bx) \\
\vdots & \\
\text{T}_{l^+}(\text{U}^l) & = \text{U}_l
\end{align*}
\]

There is also a Russell-style version.
Rules for cumulativity (optional)

Cumulativity operation:

\[
\begin{align*}
l \leq m & \quad A : U_l \\
T^m_l(A) : U_m 
\end{align*}
\]

where \( l \leq m \) means \( m = l \lor m \).

\[
T_m(T^m_l(A)) = T_l(A)
\]

We add \( T^m_l(A) = A \) if \( l = m \) and

\[
l \leq m \leq n \\
T^n_m(T^m_l(A)) = T^n_l(A)
\]

Simplifies in Russell-style.
No first universe?

We do not have $U_0$ in the theory. No constant universes, all types mentioning universes are universe polymorphic. Escardó has made an alternative experimental version of his lecture notes without $U_0$, where, e.g.,

$$N : U_0$$

is replaced by

$$N : \{U : \text{Universe}\} \to U.$$  

This turned out to be possible, but non-trivial.
Adding level-indexed products

In Agda, Level is a type so we can form level-indexed Π-types. Here it is not, so we extend the system:

\[
\begin{align*}
A \text{ type} & \quad (\alpha \text{ Level}) \\
\hline
[\alpha]A & \text{ type} \\
\end{align*}
\]

For example, we can now express the theorem that univalence of arbitrary level implies function extensionality for functions between universes of arbitrary levels:

\[
([\alpha]\text{IsUnivalent } U_\alpha) \rightarrow [\beta][\gamma]\text{FunExt } U_\beta U_\gamma
\]

Level-instantiation and level-abstraction:

\[
\begin{align*}
t : [\alpha]A & \quad l \text{ Level} \\
\hline
t l : A(l/\alpha) & \\
\end{align*}
\]

\[
\begin{align*}
u : A & \quad (\alpha \text{ Level}) \\
\hline
\langle \alpha \rangle u : [\alpha]A & \\
\end{align*}
\]

with the β-conversion rule

\[
(\langle \alpha \rangle u) l = u(l/\alpha)
\]

and the η-conversion rule.
An example using level constraints

The type

\[(A : \text{U}_l) \rightarrow (B : \text{U}_m) \rightarrow (C : \text{U}_n)\]
\[\rightarrow \text{IdU}_{l \lor m} (A \times B) (C \times A) \rightarrow \text{IdU}_{m \lor l} (B \times A) (C \times A)\]

is well-formed provided \(l \lor m = n \lor l\) and \(m \lor l = n \lor l\). No single most general solution, but three maximum points:

\[l = \alpha, m = \beta, n = \alpha \lor \beta\]
\[l = \alpha, m = \gamma \lor \alpha, n = \gamma\]
\[l = \beta \lor \gamma, m = \beta, n = \gamma\]

Cf Berry’s Gustave function.
A theory with level constraints

A *constraint* is an equation

\[ l = m \]

where \( l \) and \( m \) are level expressions.

Add new judgment form

\[ \psi (\Gamma) \]

which expresses that the constraint set \( \psi \) holds in \( \Gamma \).

We have constraint assumptions

\[ \Gamma, \psi \]

where \( \psi \) is a finite *consistent* set of constraints in \( \Gamma \).
Examples of judgments with level constraints

\[ \alpha^+ \leq \beta^+ \ (\alpha \beta \text{ Level, } \alpha \leq \beta) \]

\[ T_{\alpha^+}^\beta (U^\alpha) : U^\beta \ (\alpha \beta \text{ Level, } \alpha^+ \leq \beta) \]

The example with a non-unique most general type becomes

\[ (A : U^\alpha) \rightarrow (B : U^\beta) \rightarrow (C : U^\gamma) \]

\[ \rightarrow \text{Id}_{U^{\alpha \vee \beta}} (A \times^{\alpha \vee \beta} B) (C \times^{\gamma \vee \alpha} A) \rightarrow \text{Id}_{U^{\beta \vee \alpha}} (B \times^{\beta \vee \alpha} A) (C \times^{\gamma \vee \alpha} A) \]

in a context with constraints \( \alpha \vee \beta = \gamma \vee \alpha, \beta \vee \alpha = \gamma \vee \alpha \)
Adding constraint products

We introduce a new product operation

\[
\begin{align*}
  t : A (\psi) \\
  \quad \Rightarrow \\
  t : [\psi]A
\end{align*}
\]

with the elimination rule

\[
\begin{align*}
  t : [\psi]A \\
  \quad \quad \psi \\
  \quad \Rightarrow \\
  t : A
\end{align*}
\]

We use a term of type \([\psi]A\) only if all constraints in \(\psi\) hold.
An algorithm for solvability of the constraint set

- Bezem and Coquand 2021 give a constructive proof that either
  - there is a loop (a level expression $l$ with $l = l^+$)
  - or there is a model in $(\mathbb{N}, \text{max}, \text{succ})$.
- Sozeau is experimenting with implementations of (refinements of) the algorithm implicit in this proof.