Type Theories with Universe Level Judgments

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Universe polymorphism and univalent mathematics

Universe polymorphism (Huet 1987, Harper-Pollack 1991). Some starting points for our discussions:

- Escardó's universe polymorphic development in *Introduction to Univalent Foundations of Mathematics with Agda* `www.cs.bham.ac.uk/~mhe/HoTT-UF-in-Agda-Lecture-Notes/`
- Voevodsky 2014: *Universe polymorphic type system* - a system with constraints on universe levels. See also Courant's *EPECC* 2002 and Herbelin's *algebraic universes* 2014.
- An algorithm for solving such constraints (Bezem and Coquand 2021, Bezem, Nieuwenhuis, and Rodriguez 2008: *The Max-Atom Problem and Its Relevance*)

Looking for a simple foundation for universe polymorphism, using ideas from Coq, Agda, and Voevodsky.

The universe polymorphism of univalence

From *Introduction to Univalent Foundations of Mathematics with Agda*:

$$\begin{aligned}
&\mathrm{IdEq} : \{\mathsf{U} : \mathrm{Universe}\} \to (X\,Y : \mathsf{U}^{\cdot}) \to X \equiv Y \to X \simeq Y \\
&\mathrm{IdEq}\,X\,X\,(\mathrm{refl}\,X) = \mathrm{idsimeq}\,X \\
&\mathrm{isunivalent} : (\mathsf{U} : \mathrm{Universe}) \to \mathsf{U}^{+\cdot} \\
&\mathrm{isunivalent}\,\mathsf{U} = (X\,Y : \mathsf{U}^{\cdot}) \to \mathrm{isequiv}(\mathrm{IdEq}\,X\,Y)
\end{aligned}$$

Agda's system of universes

- An ω-tower of universes à la Russell

$$\mathrm{Set}_0 : \mathrm{Set}_1 : \mathrm{Set}_2 : \cdots$$

- A special *type* Level : Set_0 of universe levels. Universe polymorphism by quantification over Level.
- Operations $\mathrm{lzero}, \mathrm{lsucc}$, and lsup on levels.
- A "kind" Set_ω which is *not* a type. Moreover,

  Set_ω *is itself the first step in another infinite hierarchy* $\mathrm{Set}_{\omega_i} : \mathrm{Set}_{\omega_{i+1}}$. *However, this hierarchy does not support universe polymorphism, i.e. there are no sorts* $\mathrm{Set}_\omega\, l$ *for* l : Level.

A theory with level judgments

We introduce new (hypothetical) judgment forms

$$l \text{ Level} \qquad l = m$$

Contexts can contain level variables

$$\alpha \text{ Level}$$

Some rules:

$$\frac{}{\alpha \text{ Level } (\Gamma)}(\alpha\ in\ \Gamma) \qquad \frac{l \text{ Level}}{l^+ \text{ Level}} \qquad \frac{l \text{ Level} \quad m \text{ Level}}{l \vee m \text{ Level}}$$

Equations for a $\vee$-semilattice with an inflationary endomorphism $^+$:

$$\begin{aligned} l \vee l^+ &= l^+ \\ (l \vee m)^+ &= l^+ \vee m^+ \end{aligned}$$

No 0-level.

Rules for level-indexed universes (Tarski-style)

Formation rules

$$\frac{l \text{ Level}}{\mathsf{U}_l \text{ type}} \qquad \frac{A : \mathsf{U}_l}{\mathsf{T}_l(A) \text{ type}}$$

Introduction rules

$$\frac{A : \mathsf{U}_l \qquad B : \mathsf{T}_l(A) \to \mathsf{U}_m}{\Pi^{l,m} A B : \mathsf{U}_{l \vee m}} \qquad \cdots \qquad \mathsf{U}^l : \mathsf{U}_{l^+}$$

Conversion rules

$$\begin{array}{rcl} \mathsf{T}_{l \vee m}\,(\Pi^{l,m} A B) & = & \Pi(x : \mathsf{T}_l(A))\mathsf{T}_m(B\,x) \\ & \vdots & \\ \mathsf{T}_{l^+}(\mathsf{U}^l) & = & \mathsf{U}_l \end{array}$$

There is also a Russell-style version.

Rules for cumulativity (optional)

Cumulativity operation:

$$\frac{l \leqslant m \quad A : \mathsf{U}_l}{\mathsf{T}_l^m(A) : \mathsf{U}_m}$$

where $l \leqslant m$ means $m = l \vee m$.

$$\mathsf{T}_m(\mathsf{T}_l^m(A)) = \mathsf{T}_l(A)$$

We add $\mathsf{T}_l^m(A) = A$ if $l = m$ and

$$\frac{l \leqslant m \leqslant n}{\mathsf{T}_m^n(\mathsf{T}_l^m(A)) = \mathsf{T}_l^n(A)}$$

Simplifies in Russell-style.

No first universe?

We do not have U_0 in the theory. No constant universes, all types mentioning universes are universe polymorphic. Escardó has made an alternative experimental version of his lecture notes without U_0, where, e g

$$N : U_0$$

is replaced by

$$N : \{U : \mathrm{Universe}\} \to U^{\cdot}$$

This turned out to be possible, but non-trivial.

Adding level-indexed products

In Agda, Level is a type so we can form level-indexed Π-types. Here it is not, so we extend the system:

$$\frac{A \text{ type } (\alpha \text{ Level})}{[\alpha]A \text{ type}}$$

For example, we can now express the theorem that univalence of arbitrary level implies function extensionality for functions between universes of arbitrary levels:

$$([\alpha]\mathsf{IsUnivalent}\,\mathsf{U}_\alpha) \to [\beta][\gamma]\mathsf{FunExt}\,\mathsf{U}_\beta\,\mathsf{U}_\gamma$$

Level-instantiation and level-abstraction:

$$\frac{t : [\alpha]A \qquad l \text{ Level}}{t\ l : A(l/\alpha)} \qquad \frac{u : A \ (\alpha \text{ Level})}{\langle\alpha\rangle u : [\alpha]A}$$

with the β-conversion rule

$$(\langle\alpha\rangle u)\ l = u(l/\alpha)$$

and the η-conversion rule.

An example using level constraints

The type

$$(A : \mathsf{U}_l) \to (B : \mathsf{U}_m) \to (C : \mathsf{U}_n)$$

$$\to \mathsf{Id}\,\mathsf{U}_{l \vee m}\,(A \times B)\,(C \times A) \to \mathsf{Id}\,\mathsf{U}_{m \vee l}\,(B \times A)\,(C \times A)$$

is well-formed provided $l \vee m = n \vee l$ and $m \vee l = n \vee l$. No single most general solution, but three maximum points:

$$l = \alpha, m = \beta, n = \alpha \vee \beta$$

$$l = \alpha, m = \gamma \vee \alpha, n = \gamma$$

$$l = \beta \vee \gamma, m = \beta, n = \gamma$$

Cf Berry's Gustave function.

A theory with level constraints

A *constraint* is an equation

$$l = m$$

where l and m are level expression.
Add new judgment form

$$\psi\ (\Gamma)$$

which expresses that the constraint set ψ holds in Γ.
We have constraint assumptions

$$\Gamma, \psi$$

where ψ is a finite *consistent* set of constraints in Γ.

Examples of judgments with level constraints

$$\alpha^+ \leqslant \beta^+ \ (\alpha\ \beta\ \mathsf{Level}, \alpha \leqslant \beta)$$

$$T^{\beta}_{\alpha^+}(\mathsf{U}^{\alpha}) : \mathsf{U}_{\beta} \ (\alpha\ \beta\ \mathsf{Level}, \alpha^+ \leqslant \beta)$$

The example with a non-unique most general type becomes

$$(A : \mathsf{U}_{\alpha}) \to (B : \mathsf{U}_{\beta}) \to (C : \mathsf{U}_{\gamma})$$

$$\to \mathsf{Id}\,\mathsf{U}_{\alpha\vee\beta}\,(A \times^{\alpha\vee\beta} B)\,(C \times^{\gamma\vee\alpha} A) \to \mathsf{Id}\,\mathsf{U}_{\beta\vee\alpha}\,(B \times^{\beta\vee\alpha} A)\,(C \times^{\gamma\vee\alpha} A)$$

in a context with constraints $\alpha \vee \beta = \gamma \vee \alpha, \beta \vee \alpha = \gamma \vee \alpha$

Adding constraint products

We introduce a new product operation

$$\frac{t : A \ (\psi)}{t : [\psi]A}$$

with the elimination rule

$$\frac{t : [\psi]A \qquad \psi}{t : A}$$

We use a term of type $[\psi]A$ only if all constraints in ψ hold.

An algorithm for solvability of the constraint set

- Bezem and Coquand 2021 give a constructive proof that either
  - there is a loop (a level expression l with $l = l^+$)
  - or there is a model in $(\mathsf{N}, \max, \mathrm{succ})$.
- Sozeau is experimenting with implementations of (refinements of) the algorithm implicit in this proof.