Flexible presentations of graded monads

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Example: nondeterminism with backtracking and cut

These computations satisfy some equations:

\[ \text{or}(x,y) \equiv x \quad \text{whenever } x \text{ definitely cuts} \]
Models of effects from presentations

1. Effects can be modelled using monads \cite{Moggi89}.
2. which often come from presentations \cite{PlotkinPower02}.
3. which induce algebraic operations \cite{PlotkinPower03}.

Example: (based on \cite{PirógStaton17})

1. Nondeterminism with can be modelled using a monad $\text{Cut}$

\[
\text{Cut}X = \text{List}X \times \{\text{cut}, \text{nocut}\}
\]

2. which comes from the presentation of monoids with a left zero:

\[
\text{or} : 2 \quad \text{fail} : 0 \quad \text{cut} : 0 \\
\text{or}(\text{or}(x, y), z) = \text{or}(x, \text{or}(y, z)) \quad \text{or}(\text{fail}, x) = x = \text{or}(x, \text{fail}) \quad \text{or}(\text{cut}, x) = x
\]

3. which induces algebraic operations

\[
\text{or}_X : \text{Cut}X \times \text{Cut}X \rightarrow \text{Cut}X \\
\text{fail}_X : 1 \rightarrow \text{Cut}X \\
\text{cut}_X : 1 \rightarrow \text{Cut}X
\]

Example: grading nondeterminism with backtracking and cut

\[ \text{or}(x, y) \equiv y \quad \text{whenever } x \text{ has grade } \bot \]

Assign grades \( e \in \{\bot, 1, \top\} \) to computations:

\[ \begin{align*}
\top & \quad \text{don’t know anything} \\
\forall l & \quad \text{definitely cuts} \\
1 & \quad \text{or returns something} \\
\forall l & \quad \text{definitely cuts} \\
\bot & \quad \text{definitely cuts}
\end{align*} \]

Graded monad \textit{Cut}:

\[ \text{Cut}Xe = \{(xs, c) \in \text{List}X \times \{\text{cut, nocut}\} \mid (e = \bot \Rightarrow c = \text{cut}) \land (e = 1 \Rightarrow c = \text{cut} \lor xs \neq [])\} \]

Kleisli extension:

\[ f : X \to \text{Cut}Ye \]

\[ f^\dagger : \text{Cut}Xd \to \text{Cut}Y(d \cdot e) \quad \text{where} \quad 1 \cdot e = e \quad \bot \cdot e = \bot \]
Example: grading nondeterminism with backtracking and cut

1. Nondeterminism with cut can be modelled using a graded monad $\text{Cut}$

$$\text{Cut}\mathcal{X}e = \{(xs, c) \in \text{List}\mathcal{X} \times \{\text{cut, nocut}\}$$

$$\mid (e = \bot \Rightarrow c = \text{cut}) \wedge (e = 1 \Rightarrow c = \text{cut} \lor xs \neq [])\}$$

2. which comes from a graded presentation of monoids with a left zero?
3. which induces graded algebraic operations?

$$\text{or}_{d_1,d_2,\mathcal{X}} : \text{Cut}\mathcal{X}d_1 \times \text{Cut}\mathcal{X}d_2 \rightarrow \text{Cut}\mathcal{X}(d_1 \sqcap d_2) \quad (d_1, d_2 \in \{\bot, 1, \top\})$$

$$\text{fail}_\mathcal{X} : 1 \rightarrow \text{Cut}\mathcal{X}\top \quad \text{cut}_\mathcal{X} : 1 \rightarrow \text{Cut}\mathcal{X}\bot$$

The existing notions of graded presentation are not general enough

[Smirnov '08, Milius et al. '15, Dorsch et al. '19, Kura '20]
This work

Develop a notion of *flexibly graded presentation*

- Every flexibly graded presentation \((\Sigma, E)\) induces
  - a canonical graded monad \(T_{(\Sigma, E)}\)
  - along with a *flexibly graded algebraic operation* for each operation of the presentation

- Examples like Cut have computationally natural flexibly graded presentations

- The constructions are mathematically justified by locally graded categories, and a notion of *flexibly graded abstract clone*
Flexibly graded presentations

Given an ordered monoid \((E, \leq, 1, \cdot)\) of grades, a flexibly graded presentation \((\Sigma, E)\) consists of

- a signature \(\Sigma\): sets
  \[\Sigma(d'_1, \ldots, d'_n; d)\]
  of operations
  \[e \in E \quad \Gamma \vdash t_1 : d'_1 \cdot e \quad \cdots \quad \Gamma \vdash t_n : d'_n \cdot e\]
  \[\Gamma \vdash \text{op}(e; t_1, \ldots, t_n) : d \cdot e\]

- a collection of axioms \(E\): sets
  \[E(d'_1, \ldots, d'_n; d)\]
  of equations
  \[x_1 : d'_1, \ldots, x_n : d'_n \vdash t \equiv u : d\]

Part of the presentation of nondeterminism with cut:

grades \(E = \{\bot \leq 1 \leq \top\}\)

\[
\begin{align*}
\Gamma \vdash t_1 : d'_1 \cdot e & \quad \Gamma \vdash t_2 : d'_2 \cdot e \\
\Gamma \vdash \text{or}_{d'_1, d'_2}(e; t_1, t_2) : (d'_1 \sqcap d'_2) \cdot e \\
\end{align*}
\]

\[\text{or}_{\bot, e}(1; x, y) \equiv x\]
Semantics

For every flexibly graded presentation \((\Sigma, E)\), there is:

- a notion of \((\Sigma, E)\)-algebra, forming a locally graded category \(\text{Alg}(\Sigma, E)\)
- a sound and complete equational logic
- a graded monad \(T_{(\Sigma, E)}\) on \(\text{Set}\) and concrete functor \(R_{(\Sigma, E)} : \text{Alg}(\Sigma, E) \to \text{EM}(T_{(\Sigma, E)})\), satisfying a universal property

\[
\begin{array}{ccc}
\text{Alg}(\Sigma, E) & \xrightarrow{R_{(\Sigma, E)}} & \text{EM}(T_{(\Sigma, E)}) \\
& \downarrow{R'} & \downarrow{\text{EM}(\alpha)} \\
& \text{EM}(T') & \xrightarrow{\alpha} \text{T'}
\end{array}
\]

- for every op in \(\Sigma\), a flexibly graded algebraic operation for \(T_{(\Sigma, E)}\)

A large class of graded monads have flexibly graded presentations:

- exactly the graded monads on \(\text{Set}\) that preserve conical sifted colimits

[Wood '76]