Higher-Order Universe Operators in Martin-Löf Type Theory with One Mahlo Universe

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MLTT with One Mahlo Universe

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- Setzer [2, 3] introduced MLTT with one Mahlo universe (called $\mathbf{MLM}$), and determined its proof-theoretic ordinal
  - Mahlo universes have a reflection property similar to the ones of weakly Mahlo cardinals and recursively Mahlo ordinals
- Later, Rathjen [4] formulated the Mahlo property in constructive set theory $\mathbf{CZF}$, and showed that $\mathbf{CZF}$ with an axiom asserting the existence of a Mahlo set is interpretable in Setzer's extension of MLTT

MLTT with One Mahlo Universe

- We use
  - the logical framework adopted in Agda
  - the families-of-sets formulation of Mahlo universes
- a Mahlo universe type $\mathrm{V} : \mathrm{Set}$ with the decoding function $\mathrm{T} : \mathrm{V} \to \mathrm{Set}$ "reflects" any operator on families of sets in V:

$$\frac{\Gamma \vdash f : (\sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{V}) \to (\sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{V})}{\Gamma \vdash \widehat{\mathrm{U}}_f : \mathrm{V}} \; \widehat{\mathrm{U}}\text{-}I$$

$$\mathrm{T}\, \widehat{\mathrm{U}}_f = \mathrm{U}_f$$

U_f is a subuniverse of V with the decoding function $\widehat{\mathrm{T}}_f : \mathrm{U}_f \to \mathrm{V}$ such that $(\mathrm{U}_f, \mathrm{T}_f)$ is closed under f

MLTT with One Mahlo Universe

- For a given $f : (\sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V}) \to (\sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V})$, the closedness of $(\mathrm{U}_f, \mathrm{T}_f)$ is expressed by Res^i_f with $\mathrm{T}_f := \mathrm{T} \circ \widehat{\mathrm{T}}_f$:

$$\mathsf{Res}^1_f : (\sum_{(a:\mathrm{U}_f)} \mathrm{T}_f a \to \mathrm{U}_f) \to \mathrm{U}_f$$

$$\mathsf{Res}^2_f : (c : \sum_{(a:\mathrm{U}_f)} \mathrm{T}_f a \to \mathrm{U}_f) \to \mathrm{T}_f(\mathsf{Res}^1_f\, c) \to \mathrm{U}_f$$

$$\widehat{\mathrm{T}}_f(\mathsf{Res}^1_f\, c) = \mathsf{p}_1\Big(f(\widehat{\mathrm{T}}_f(\mathsf{p}_1\, c), \lambda x.\widehat{\mathrm{T}}_f(\mathsf{p}_2\, c\, x))\Big)$$

$$\widehat{\mathrm{T}}_f(\mathsf{Res}^2_f\, c\, d) = \mathsf{p}_2\Big(f(\widehat{\mathrm{T}}_f(\mathsf{p}_1\, c), \lambda x.\widehat{\mathrm{T}}_f(\mathsf{p}_2\, c\, x))\Big)\, d$$

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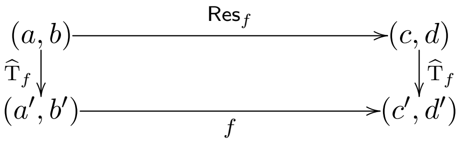
$$\frac{\underline{\sum_{(a:\mathrm{U}_f)} \mathrm{T}_f a \to \mathrm{U}_f}}{\underline{\sum_{(a:\mathrm{V})} \mathrm{T}\, a \to \mathrm{V}}}$$



Examples of Subuniverses of a Mahlo Universe

- A universe U above $a : \mathrm{V}$ and $b : \mathrm{T}\, a \to \mathrm{V}$:
  define $f_0 : (\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V}) \to (\sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V})$ as $f_0 := \lambda c.(a, b)$ where c does not occur in a nor b freely, then we have $(\widehat{\mathrm{U}}_{f_0}, \widehat{\mathrm{T}}_{f_0})$

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\begin{aligned}
\widehat{\mathrm{T}}_{f_0}(\mathsf{Res}^1_{f_0}(\widehat{\mathrm{N}}_{0,f_0}, \lambda x.\mathsf{C}_0 x)) &= \mathsf{p}_1(f_0(\widehat{\mathrm{T}}_{f_0}(\widehat{\mathrm{N}}_{0,f_0}), \lambda x.\widehat{\mathrm{T}}_{f_0}(\mathsf{C}_0 x))) \\
&= \mathsf{p}_1((\lambda c.(a, b))(\widehat{\mathrm{N}}_0, \lambda x.\widehat{\mathrm{T}}_{f_0}(\mathsf{C}_0 x))) \\
&= a.
\end{aligned}
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\end{aligned}$$

- A super universe: by reflecting
  $y : \sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V} \vdash \lambda c.y : (\sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V}) \to (\sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V})$,
  we have $(\widehat{\mathrm{U}}_f, \widehat{\mathrm{T}}_f)$ with $f := \lambda c.y$, so put

$$g := \lambda y.(\widehat{\mathrm{U}}_f, \widehat{\mathrm{T}}_f) : (\sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V}) \to (\sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V})$$

  then we obtain a super universe $\widehat{\mathrm{U}}_g$ because g is a universe operator

Another Extension with Universes: The System $\mathbf{MLQ}$

- Rathjen, Griffor and Palmgren [5] introduced the system $\mathbf{MLQ}$ of MLTT, which includes the type Q
  - Q is an inductive type of codes for operators which provides universes closed under the universe operators constructed previously

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- In sum, both $\mathbf{MLM}$ and $\mathbf{MLQ}$ were introduced to study the type-theoretic counterparts of large sets
- The comparison in terms of proof-theoretic strength was already attempted in the literature (e.g., [9, 10, 3])
  We aim to investigate the relationship more directly by simulating $\mathbf{MLQ}$ in $\mathbf{MLM}$

Operators on Mahlo Universe V

- Inspired by [9], we define operators of order n and families of order n on V:

$$
\begin{aligned}
\mathrm{O}_0 &:= \mathrm{V} \\
\mathrm{F}_n &:= \sum_{(a:\mathrm{V})} \mathrm{T}\,a \to \mathrm{O}_n && \text{(families of operators of order } n\text{)} \\
\mathrm{O}_{n+1} &:= \mathrm{F}_n \to \mathrm{F}_n && \text{(operators of order } n+1\text{)}
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- In this talk, we use

$$\mathrm{O}_0 = \mathrm{V} \qquad \mathrm{F}_0 = \sum_{(a:\mathrm{V})} \mathrm{T}\,a \to \mathrm{O}_0$$

$$\mathrm{O}_1 = \mathrm{F}_0 \to \mathrm{F}_0 \qquad \mathrm{F}_1 = \sum_{(a:\mathrm{V})} \mathrm{T}\,a \to \mathrm{O}_1$$

  only

Simulation of $\mathbf{MLQ}$ in $\mathbf{MLM}$

- An element of Q should be simulated with $u : \mathrm{F}_1 \to \mathrm{O}_1$ in $\mathbf{MLM}$ s.t. for any $(a_1, b_1) : \mathrm{F}_1$, $u\,(a_1, b_1) : \mathrm{O}_1$ is a universe operator which takes $y : \sum_{(x:\mathrm{V})} \mathrm{T}\,x \to \mathrm{V}$ and returns a universe $\widetilde{\mathrm{U}}$ satisfying
  - $\widetilde{\mathrm{U}}$ contains y
  - $\widetilde{\mathrm{U}}$ is closed under all operators in the family $(a_1, b_1) : \mathrm{F}_1$

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  - $\widetilde{\mathrm{U}}$ contains y
  - $\widetilde{\mathrm{U}}$ is closed under all operators in the family $(a_1, b_1) : \mathrm{F}_1$
- We construct such a $\widetilde{\mathrm{U}}$ with a family of subuniverses of V:

$$\widetilde{\mathrm{U}} := \widehat{\Sigma}\, a_1\, (\lambda e.\widehat{\Sigma}\, \widehat{\mathrm{N}}_2\, (\lambda z.\widehat{\mathrm{U}}_h)) \text{ with } h := \mathsf{C}_2\, (\lambda c.y)\, (b_1\, e)\, z : \mathrm{O}_1$$

$$\textstyle\sum_{(e:\mathrm{T}\,a_1)} \sum_{(z:\mathrm{N}_2)} \mathrm{U}_{\mathsf{C}_2\, (\lambda c.y)\, (b_1\, e)\, z}$$

The point is twofold: the use of the eliminator C_2 / the use of a parameter of greater order

Higher-Order Universe Operators

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Higher-Order Universe Operators

- Palmgren [9] introduced a family $\mathbf{ML}^n$ of systems of higher-order universe operators, and showed that $\mathbf{MLQ}$ is an instance of $\mathbf{ML}^3$
- E.g., $\mathbf{ML}^4$ has a third-order universe operator which enables to construct second-order universe operators inductively
  - On the other hand, $\mathbf{ML}^3$ (hence $\mathbf{MLQ}$) has no third-order universe operator and cannot construct a second-order universe operator newly

Simulation of ML^4 in MLM

- A case study: simulate a third-order operator o_3 s.t. if o_3 takes a second-order operator o_2, a first-order operator o_1 and $y : \sum_{(x:\mathrm{V})} \mathrm{T}\, x \to \mathrm{V}$, then o_3 returns a universe $\widetilde{\mathrm{U}}$ satisfying
  - $\widetilde{\mathrm{U}}$ contains y
  - $\widetilde{\mathrm{U}}$ is closed under o_1
  - $\widetilde{\mathrm{U}}$ is closed under $o_2\, o_1$, i.e., the first-order operator obtained by applying o_2 to o_1

- In this case, we simulate $\widetilde{\mathrm{U}}$ by the following family:

$$\sum\nolimits_{(z:\mathrm{N}_3)} \mathrm{U}_{\mathsf{C}_3}\, (\lambda c.y)\; o_1\; (o_2\; o_1)\; z$$

Here we use a higher-order parameter to simulate $\widetilde{\mathrm{U}}$

Concluding Remarks

- We have sketched how to simulate higher-order universe operators of $\mathbf{MLQ}$ and $\mathbf{ML}^n$ in $\mathbf{MLM}$
- On the other hand, Dybjer and Setzer showed that
  - $\mathbf{ML}^n$ can be defined by induction-recursion [11]
  - a version of Mahlo universes can be defined by induction-recursion as well [12]

  So induction-recursion is more comprehensive than Mahlo universes in this respect

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    - a version of Mahlo universes can be defined by induction-recursion as well [12]

  So induction-recursion is more comprehensive than Mahlo universes in this respect
- But we believe that Mahlo universes are still to be investigated further, since they can be extended in a natural way (hyper Mahlo universes, autonomous Mahlo universes, and so on, see [13])
    - These extensions should enable us to study the type-theoretic formulation of large sets further

Thank you for your attention!

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