Extending truth table natural deduction to predicate logic

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Natural deduction rules from truth tables

Earlier work: derive natural deduction rules for a connective c from its truth table definition.
Summarizing:

- rules have a generic format, allowing a general proof-theoretic study.
- produces both the classical and constructive derivation rules for standard connectives (and also for less standard connectives).
- "good" properties: proof normalization, subformula property, general Kripke semantics (sound and complete).
- we can study connectives "in isolation", e.g. from the classic rules for $\to$ one can derive Peirce's Law.

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- "good" properties: proof normalization, subformula property, general Kripke semantics (sound and complete).
- we can study connectives "in isolation", e.g. from the classic rules for $\to$ one can derive Peirce's Law.
- for monotone connectives (like $\wedge, \vee$), the classical and constructice rules are equivalent; for non-monotonic connectives (like $\to$, $\neg$) this is not the case.
- One classical non-monotonic connective makes all non-monotonic connectives classical.

General format of rules

This is the general format of rules for proposition logic, where Γ and φ are auxiliary and $A_1, \ldots, A_n, B_1, \ldots B_m$ are rule specific.

$$\frac{\Gamma \vdash A_1 \quad \ldots \quad \Gamma \vdash A_n \qquad \Gamma, B_1 \vdash \varphi \quad \ldots \quad \Gamma, B_m \vdash \varphi}{\Gamma \vdash \varphi}$$

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Examples:

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash \varphi \qquad \Gamma, B \vdash \varphi}{\Gamma \vdash \varphi} \vee\text{-el}$$

$$\frac{\Gamma \vdash A \wedge B \qquad \Gamma, A \vdash \varphi}{\Gamma \vdash \varphi} \wedge\text{-el}$$

Extending with rules for predicate logic

We want

1. simple generic rules, preferably extensible to other quantifiers.
2. constructive rules as a simple variation of the classical ones.
3. study quantifiers and their rules in isolation.
4. proof theoretic properties: (Kripke) semantics, soundness and completeness, proof normalization, subformula property.

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NB. In sequent calculus one moves to multi-conclusion judgments. This can also be done in natural deduction, but we want to refrain from that and stay close to the spirit of natural deduction.

Some more examples

Here are some more classically provable judgments from predicate logic (not constructively provable) that we can deal with.

- $\exists x(D\,x \to \forall y.D\,y)$
  Has a normal proof (satisfying the subformula property) using constructive $\to$ and classical $\forall$.

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Our rules will achieve the above. We assume that domains are non-empty. (This can be relaxed, by adapting the rules.)

- The rules we get for constructive $\forall, \exists$ are equivalent to the standard ones.
- The rules we get for classical $\forall, \exists$ are stronger, but do not imply the rules for classical proposition logic.

The formulas and the form of the rules

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- Classical intuition of a witness constant $a_{\forall x.\psi}$:
  - if $\forall x.\psi$ holds, $a_{\forall x.\psi}$ is an arbitrary element
  - if not $\forall x.\psi$, $a_{\forall x.\psi}$ is some element d such that $\neg\psi[d/x]$.

  (Similarly for $a_{\exists x.\psi}$.)
- In the classical semantics, the interpretation of witness constants is exacly that: $a_{\forall x.\psi}$ is interpreted as an element d such that $\psi[d/x]$ has the same interpretation as $\forall x.\psi$.

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- Constructively, the interpretation of witness constants is just the local fresh parameter used standardly in deduction rules

The rules (I)

$\exists$-elimination

- Classical $\exists$-elimination:

$$\frac{\Gamma \vdash \exists y.\varphi \quad \Gamma, \varphi[a_{\exists y.\varphi}/y] \vdash \psi}{\Gamma \vdash \psi}\ \exists\text{-el}$$

NB! No freshness side condition!

∃-elimination

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$$\frac{\Gamma \vdash \exists y.\varphi \quad \Gamma, \varphi[a_{\exists y.\varphi}/y] \vdash \psi}{\Gamma \vdash \psi} \exists\text{-el}$$

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- Constructive ∃-elimination:

$$\frac{\Gamma; A \vdash \exists y.\varphi \quad \Gamma, \varphi[a_{\exists y.\varphi}/y] \vdash \psi}{\Gamma; A \vdash \psi} \exists\text{-el}$$

  Restriction: $a_{\exists y.\varphi}$ should not occur in Γ or ψ.
  This avoids e.g. deriving $\exists y.P\, y \vdash P\, a_{\exists y.\varphi}$.

The rules (II)

$\forall$-introduction

- Classical $\forall$-introduction (for $\forall y.\varphi$)

$$\frac{\Gamma \vdash \varphi[a_{\forall y.\varphi}/y]}{\Gamma \vdash \forall y.\varphi}\ \forall\text{-in}$$

- Constructive $\forall$-introduction (for $\forall y.\varphi$)

$$\frac{\Gamma \vdash \varphi[a_{\forall y.\varphi}/y]}{\Gamma \vdash \forall y.\varphi}\ \forall\text{-in}$$

Restriction: $a_{\forall y.\varphi}$ should not occur in Γ.

Classical and constructive ∃-intro and ∀-elim are standard (t is an arbitrary closed term)

$$\frac{\Gamma \vdash \varphi[t/y]}{\Gamma \vdash \exists y.\varphi}\ \exists\text{-in} \qquad\qquad \frac{\Gamma \vdash \forall y.\varphi}{\Gamma \vdash \varphi[t/y]}\ \forall\text{-el}$$

A derivation of

$$\forall x.(B\,x \vee C) \vdash (\forall x.B\,x) \vee C$$

that satisfies the subformula property.
We abbreviate $a := a_{\forall x.(B\,x \vee C)}$ and $\Phi := \forall x.(B\,x \vee C)$ and $\Psi := (\forall x.B\,x) \vee C$.

$$\dfrac{\dfrac{\vdots}{\Phi \vdash B\,a \vee C} \qquad \dfrac{\dfrac{\Phi, B\,a \vdash B\,a}{\Phi, B\,a \vdash \forall x.B\,x}\ \forall\text{-in}}{\Phi, B\,a \vdash \Psi}\ \vee\text{-in} \qquad \dfrac{\vdots}{\Phi, C \vdash \Psi}}{\Phi \vdash \Psi}\ \vee\text{-el}$$

Conclusion and further work

Summarizing:

- We have defined "stand alone" natural deduction rules for classical (and constructive) predicate logic $\forall, \exists$.
- The constructive rules are a variation on the classical ones.
- Kripke semantics that is sound and complete.
- Classical semantics that is sound and complete.
- Derivations satisfying the subformula property for a number of well-known classically provable statements.
- The rules follow mostly the "standard form", so proof normalizations should work.

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Further work:

- Proof term interpretation and proof of normalization
- Translate to classical logic that is obtained via multi-conclusion sequents.
- Study the precise fragment constructive proposition logic + classical quantifiers.
- Extend to other quantifiers.