Cubical models are cofreely parametric

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Outline

Introduction

A first example: parametric categories

A general theory

More examples: lex categories and clans

Parametricity for type theory

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A model of type theory is parametric if:

- Any type comes with a relation.
- Any term respects these.

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Observation

Cubical structures arise when working with parametricity.

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Observation

Cubical structures arise when working with parametricity.

- A presheaf model of parametric type theory.
  [Bernardy, Coquand, Moulin 2015]
- Cubical categories for higher-dimensional parametricity.
  [Johann, Sojakova 2017]
- Internal parametricity for cubical type theory.
  [Cavallo, Harper 2020]

Cubical models are cofreely parametric

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For many:

- Notions of model of type theory.
- Variants of cubes.

Cubical models are cofreely parametric

For many:

- Notions of model of type theory.
- Variants of cubes.

There is a notion of parametricity such that:

forgetful functor

{*Models of type theory*} $\perp$ {*Parametric models*}

$\mathcal{C} \mapsto$ {*cubical types in* $\mathcal{C}$}

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Parametric categories

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Definition

A category $\mathcal{C}$ is parametric if we are given:

- An endofunctor of $\mathcal{C}$:
$$X \mapsto X_*$$
- Natural transformations:
$$d^0_X, d^1_X \quad : \quad X_* \to X$$

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Any object X comes with a relation:

$$d^0_X, d^1_X \quad : \quad X_* \to X$$

Any morphism respects these.

Categories of semi-cubical objects

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Definition

Let $\square$ be the (opposite of the) category of semi-cubes, so that:

$$\mathcal{C}^{\square} \quad = \quad \{\textit{Semi–cubical objects in } \mathcal{C}\}$$

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Proposition

$$\{\textit{Categories}\} \;\underset{\mathcal{C} \,\mapsto\, \mathcal{C}^{\square}}{\overset{\textit{forgetful functor}}{\rightleftarrows}}\; \{\textit{Parametric categories}\} \qquad \bot$$

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Example

The category $\square$ is the monoidal category generated by:

$$d^0, d^1 \quad : \quad \mathbb{I} \to 1$$

Parametric models

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Definition

A parametric model is an $\mathcal{M}$-module.

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A parametric model is an $\mathcal{M}$-module.

Example

A $\square$-module is a category $\mathcal{C}$ with a monoidal functor:

$$\alpha \quad : \quad \square \to \mathit{End}_{\mathcal{C}}$$

or equivalently:

$$\begin{array}{rcl} \_{*} & : & \mathit{End}_{\mathcal{C}} \\ d^0, d^1 & : & \mathit{Hom}_{\mathit{End}_{\mathcal{C}}}(\_{*}, 1) \end{array}$$

Cofreely parametric models

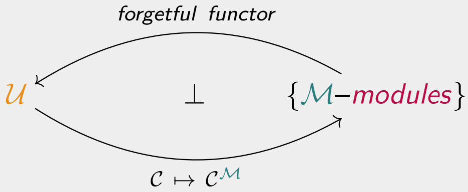
Cofreely parametric models

Theorem

$$\mathcal{U} \underset{\mathcal{C} \mapsto \mathcal{C}^{\mathcal{M}}}{\overset{\textit{forgetful functor}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \{\mathcal{M}\text{–}\textit{modules}\}$$

Cofreely parametric models

Theorem



Example

$$\begin{aligned} \mathcal{U} &= \{\textit{Categories}\} \\ \mathcal{M} &= \square \\ \{\mathcal{M}\text{–}\textit{modules}\} &= \{\textit{Parametric categories}\} \end{aligned}$$

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Notions of parametricity for categories

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Example

Categories of cubical objects are cofreely parametric,
by adding to $\square$ a morphism:

$$r \quad : \quad 1 \to \mathbb{I}$$

such that:

$$d^0 \circ r = d^1 \circ r = id$$

Variants of parametricity for lex categories and clans

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Theorem

Lex categories (or clans) form a symmetric monoidal closed category.

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Lex categories of truncated semi-cubical (or cubical) objects are cofreely parametric.

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Clans of Reedy fibrant semi-cubical (or cubical) objects are cofreely parametric.

Further work

- To work with a 1-category of models, we use strict versions of lex categories and clans.
- Models with Π-types or universes do not fit.
- Univalence and Kan cubes do not fit.

Thank you!

I'm looking for a post-doc!

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