Quantitative Perspectives on Generalized Applications

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The Calculus ΛJ

$$\frac{}{\Gamma, x : A \vdash x : A}\ \text{AX} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ \to_i$$

$$\frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A \qquad \Gamma, y : B \vdash r : C}{\Gamma \vdash t(u, y.r) : C}\ \to_e$$

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(Terms) $t, u, r \quad ::= v \mid t(u, y.r)$

(Values) $v \quad ::= x \mid \lambda x.t$

Intuition

$t(u, y.r) \rightsquigarrow$ let $y = tu$ in r or $[tu/y]r$

Motivations

The calculus ΛJ:

- Has proof-theoretical foundations.
- Combines applications and explicit substitutions.
- Has built-in sharing.
- Bears similarities to A-normal forms[1]/crumbling[2].
- Has ties to the sequent calculus.

[1] Flanagan et al. "The Essence of Compiling with Continuations".
[2] Accattoli et al. "Crumbling Abstract Machines".

State of the Art

Some previous results:

- SN of the simply typed calculus, standardization, confluence (Joachimski and Matthes);
- An idempotent intersection type system for SN (Matthes);
- Relationship between natural deduction and sequent calculus (Espírito Santó, Frade, Pinto).
- A call-by-value calculus (Espírito Santó).

Our Approach

We take an operational approach to generalized applications, guided by a resource-aware model given by quantitative types.

Main Contributions

- A CBN calculus compatible with a quantitative type system.
- CBN/CBV solvability for generalized applications.
- Logical characterizations and equivalence with other calculi.

Quantitative Types

Logical characterization

Normalizable $\iff$ typable.

Theorem (Subject reduction)

$t_1 \to t_2$ *and* $\Phi_1 = \Gamma \vdash t_1 : \tau$
$\implies \Phi_2 = \Gamma \vdash t_2 : \tau$ *and* $\mathrm{sz}(\Phi_1) > \mathrm{sz}(\Phi_2)$.

Reduction Rules

(Permutation) $t(u, x.r)(u', y.r') \to_{\pi} t(u, x.r(u', y.r))$

Reduction Rules

$$\text{(Permutation)} \quad t(u, x.r)(u', y.r') \rightarrow_{\pi} t(u, x.r(u', y.r))$$

$$\text{(CBN)} \quad (\lambda x.t)(u, y.r) \rightarrow_{\beta} \{\{u/y\}r/x\}t$$

Intuition

let y = $(\lambda x.t)u$ in $r \rightarrow_{\beta}$ let y = $\{u/x\}t$ in $r \rightarrow_{\beta} \{\{u/x\}t/y\}r$

Reduction Rules

(Permutation) $t(u, x.r)(u', y.r') \to_\pi t(u, x.r(u', y.r))$

(CBN) $(\lambda x.t)(u, y.r) \to_\beta \{\{u/y\}r/x\}t$

(CBV) $(\lambda x.t)(u, y.r) \to_{\beta_v} \{\{u \backslash\!\backslash y\}r \backslash\!\backslash x\}t$

Intuition

let y = $(\lambda x.t)u$ in $r \to_\beta$ let y = $\{u/x\}t$ in $r \to_\beta \{\{u/x\}t/y\}r$

Integrating Distance

The distant rule combines β-reduction and permutation to focus on computation.

$$\mathsf{D}\langle \lambda x.t \rangle (u, y.r) \to_{d\beta} \{\mathsf{D}\langle \{u/x\}t \rangle / y\} r \qquad \mathsf{D} ::= \Diamond \mid t(u, x.\mathsf{D})$$

Distance is not Based on π

Rule π is rejected by the quantitative type system. We use a different rule p2:

$$t(u, y.\lambda x.r) \to_{\mathsf{p2}} \lambda x.t(u, y.r)$$

Compare:

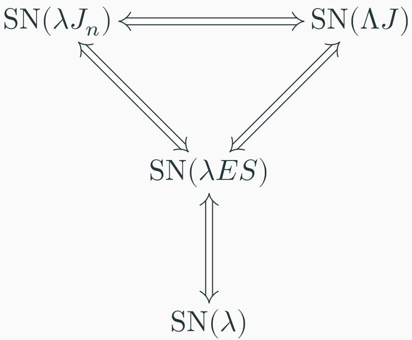
(Based on p2) $\mathsf{D}\langle \lambda x.t \rangle (u, y.r) \to \{\mathsf{D}\langle \{u/x\}t \rangle / y\} r$

CBN-like rule: duplication or easure of D

(Based on π) $\mathsf{D}\langle \lambda x.t \rangle (u, y.r) \to \mathsf{D}\langle \{\{u/x\}t/y\} r \rangle$

CBV-like rule: sharing of D

Preservation of Strong Normalization



Solvable Characterizations

Theorem (Operational characterization)

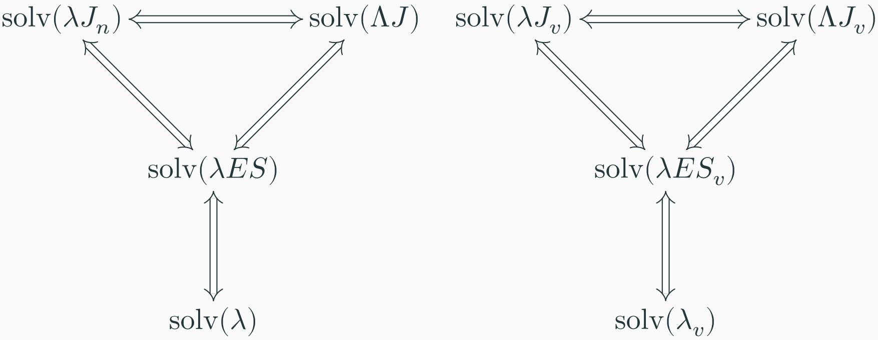
CBN/CBV solvable $\iff$ *CBN/CBV solving normalizable.*

Theorem (Logical characterization)

CBN/CBV solvable $\iff$ *typable in* $\mathcal{N}/\mathcal{V}$.

CBV solving relation: reduction under head and weak contexts.

Preservation of Solvability



(CBN) (CBV)

Three Equivalent Definitions of Solvability

Several definitions of solvability coexist, all equivalent in the λ-calculus:

1. $\exists u.\mathtt{H}\langle t\rangle \to u$, where u is fully normal.
2. $\mathtt{H}\langle t\rangle \to \mathtt{I}$,
3. $\forall u.\mathtt{H}\langle t\rangle \to u$,

Conjecture

The definitions are equivalent in ΛJ_v.

Hint: $\mathtt{I}(u, x.x) \to_{\beta v} u$ for all u.

Conclusion

Further points of interest:

- A partial genericity lemma.
- Investigating CBV solvability.
- Connection with ANF.
- Tight types.